Uniform tight frames for signal processing and communication

Peter G. Casazza*
Department of Mathematics
University of Missouri-Columbia
Columbia, MO 65211
pete@math.missouri.edu

Jelena Kovacević
Bell Labs
Murray Hill, NJ 07974
jelena@research.bell-labs.com

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Abstract

We will review the latest developements concerning uniform tight frames and their applications to signal processing.

1 Introduction

Frames are redundant sets of vectors in a Hilbert space which yield one natural representation for each vector in the space, but which may have infinitely many different representations for a given vector. Frames have been used in signal processing because of their resilience to additive noise[8], resilience to quantization[15], as well as their numerical stability of reconstruction[8], and greater freedom to capture signal characeristics[2, 3]. Recently, several new applications for (uniform tight) frames have been developed. The first, developed by Goyal and Kovačević and Vetterli[10, 11, 12, 13, 14], uses the redundancy of a frame to mitigate the effect of losses in packet-based communication systems. Modern communication networks transport packets of data from a "source" to a "recipient". These packets are sequences of information bits of a certain length surrounded by error-control, addressing, and timing information that assure that the packet is delivered without errors. It accomplishes this by not delivering the packet if it contains errors. Failures here are due primarily to

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buffer overlfows at intermediate nodes in the network. So to most users, the behavior of a packet network is not characterized by random loss, but by unpredictable transport time. This is due to a protocol, invisible to the user, that retransmits lost packets. Retransmission of packets takes much longer than the original transmission. In many applications, retransmission of lost packets is not feasible and the potential for large delay is unacceptable.

If a lost packet is independent of the other transmitted data, then the information is truly lost to the receiver. But if there are dependencies between transmitted packets, one could have partial or complete recovery despite losses. This leads one naturally to use frames for encoding. But the question is: What are the best frames for this purpose? With an additive noise model for quantization, Goyal and Kovačević[11] show that a uniform frame minimizes mean-squared error if and only if it is tight. So it is this class of frames - the uniform normalized tight frames (see Section 2 for the definitions) - which we need to identify.

Another recent important application of uniform normalized tight frames is in multiple-antenna code design[7, 17]. Much theoretical work has been done to show that communication systems that employ multiple antennas can have very high channel capacities[9, 20]. These methods rely on the assumption that the receiver knows the complex valued Rayleigh fading coefficients. To remove this assumption, in Refs. 6 and 18 new classes of unitary space-time signals are proposed. If we have N transmitter antennas and we transmit in blocks of M time samples (over which the fading coefficients are approximately constant), then a **constellation of K unitary space-time signals** is a (weighted by $\sqrt{M}$) collection of $M \times N$ complex matricies $\{\Phi_k\}$ for which $\Phi_k^* \Phi_k = I$. The n^{th} column of any ϕ_k contains the signal transmitted on antenna n as a function of time. The only structure required in general is the time-orthogonality of the signals.

Originally it was believed that designing such constellations was a too cumbersome and difficult optimization problem for practice. However, in Ref. 19 it was shown that constellations arising in a "systematic" fashion can be done with relatively little effort. Systematic here means that we need to design high-rate space-time constellations with low encoding and decoding complexity. It is known that full transmitter diversity (i.e. where the constellation is a set of unitary metrices whose differences have nonzero determinant) is a desirable property for good performance. In a *tour-de-force*, Hassibi, Hochwald, Shokrollahi, and Sweldens[17] used fixed-point-free groups and their representations to design high-rate constellations with full diversity. Moreover, they classified all full-diversity constellations that form a group, for all rates and numbers of transmitter antennas.

For these applications, and a host of other applications in signal processing, it has become important that we understand the class of uniform normalized tight frames. In this paper[6], we will make the first systematic study of this class of frames.

2 Frames

In this section we will introduce the concepts to be used in the paper.

Definition 1. *A family* $\{f_i\}_{i\in I}$ *is a* **frame** *for a Hilbert space* H *if there are constants* $0 < A, B$ *satisfying:*

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad \textit{for all } f \in H.$$

We call A, B *a set of* **frame bounds** *for the frame with* A *the* **lower frame bound** *and* B *the* **upper frame bound**. *We call* $\{f_i\}_{i\in I}$ *a* **tight frame** *if* $A = B$ *and a* **normalized tight frame** *if* $A = B = 1$. *If* $\|f_i\| = \|f_j\|$, *for all* $i, j \in I$ *this is a* **uniform frame**. *If also* $A = B = 1$, *we have a* **uniform normalized tight frame**.

For the basic theory of frames we refer the reader to Ref. 3 and 21. If $\{f_i\}_{i\in I}$ is a frame for H, we define the **synthesis operator** $T : \ell_2^I \to H$ by $Te_i = f_i$ where $\{e_i\}_{i\in I}$ is an orthonormal basis for ℓ_2^I. The **analysis operator** is the operator T^*. A direct calculation shows that

$$T^*f = \sum_{i\in I} \langle f, f_i\rangle e_i, \quad \text{for all } f \in H.$$

Hence,

$$\|T^*f\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

It follows that:

Theorem 1. *Let* $\{f_i\}_{i\in I}$ *be a sequence in a Hilbert space* H *with synthesis operator* T. *The following are equivalent:*

(1) $\{f_i\}_{i\in I}$ *is a frame for* H.

(2) T *is a bounded, linear and onto map.*

(3) T^* *is an isomorphism.*

Moreover, $\{f_i\}_{i\in I}$ *is a normalized tight frame if and only if* T *is a quotient map (or equivalently,* T^* *is a partial isometry).*

Corollary 1. *If* $\{f_i\}_{i\in I}$ *is a frame for* H *then* $S = TT^*$ *is an invertible operator on* H *called the* **frame operator**.

Now,

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

It follows that for all $f \in H$,

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

Hence, S is an invertible, self-adjoint, positive operator on H with $AI \leq S \leq BI$. So $\{f_i\}_{i\in I}$ is a normalized tight frame if and only if $S = I$ and we have **reconstruction** of any function $f \in H$ by:

$$Sf = \sum_{i\in I} \langle S^{-1}f, f_i\rangle f_i.$$

We say that two frames $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ for H are **equivalent** if there is an invertible operator L on H for which $Lf_i = g_i$ for all $i \in I$, and they are **unitarily equivalent** if L can be chosen to be a unitary operator. A direct calculation shows that $\{S^{-1/2}f_i\}$ is a normalized tight frame for any frame $\{f_i\}$. In particular, every frame is **equivalent** to a normalized tight frame.

A useful way to view normalized tight frames is due to Neumark[1] (see also Han and Larson[16]):

Theorem 2. [**Han and Larson**] *A family $\{f_i\}_{i\in I}$ in a Hilbert space H is a normalized tight frame for H if and only if there is a larger Hilbert space $H \subset K$ and an orthonormal basis $\{e_i\}_{i\in I}$ for K so that the orthogonal projection P of K onto H satisfies: $Pe_i = f_i$, for all $i \in I$.*

3 Uniform Normalized Tight Frames: Some Examples

For any natural number N, we write H_N for an N-dimensional Hilbert space. There are two general classes of uniform tight frames which are commonly used. The (general) harmonic frames and the tight Gabor frames.

Definition 2. *Fix $M \geq N$, $|c_i| = 1$, and $\{b_i\}_{i=1}^N$ with $|b_i| = \frac{1}{\sqrt{M}}$. Let $\{c_i\}_{i=1}^N$ be distinct M^{th} roots of c, and for $0 \leq k \leq M-1$, let*

$$\phi_k = (c_1^k b_1, c_2^k b_2, \cdots, c_N^k b_N).$$

Then $\{\phi_k\}_{k=0}^{M-1}$ is a uniform normalized tight frame for H_N called a **general harmonic frame**.

For the other general class, we introduce two special operators on $L^2(\mathbb{R})$. Fix $0 < a, b$ and for $f \in L^2(\mathbb{R})$ define **translation by a** as

$$T_a f(t) = f(t-a),$$

and **modulation by b** as

$$E_b f(t) = e^{2\pi imbt} f(t).$$

If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, we call it a **Gabor frame** (or a **Weyl-Heisenberg frame**). It is clear that this class of frames are uniform. Also, since the frame operator S for a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ must commute with translation and modulation, each Gabor frame is equivalent to the (uniform) normalized tight Gabor frame $\{E_{mb}T_{na}S^{-1/2}g\}_{m,n\in\mathbb{Z}}$ For an introduction to Gabor frames we refer the reader to Ref. 5 and 18.

Although we would like to classify all uniform tight frames, especially those which can be obtained by reasonable "algorithms", this is essentially an impossible task in general because every finite family of norm one vectors in a Hilbert space can be extended to become a uniform tight frame.

Theorem 3. *If $\{f_i\}_{i=1}^M$ is a family of norm one vectors in a Hilbert space H, then there is a uniform tight frame for H which contains the family $\{f_i\}_{i=1}^M$.*

Proof. For each $1 \leq i \leq M$ choose an orthonormal basis $\{g_{ij}\}_{j\in J}$ for H which contains the vector f_i. Now the family $\{g_{ij}\}_{i=1,j\in J}^M$ is made up of norm one vectors and for any $f \in H$ we have

$$\sum_{i=1}^{M}\sum_{j\in J}|\langle f, g_{ij}\rangle|^2 = \sum_{i=1}^{M}\|f\|^2 = M\|f\|^2.$$

□

In the above construction we get as a tight frame bound the number of elements in the family $\{f_i\}_{i=1}^M$. In general this is best possible. For example, just let $f_i = f_j$ for all $1 \leq i, j \leq M$.

There is another general class of uniform tight frames (see Ref. 11).

Theorem 4. *A family $\{f_i\}_{i=1}^{N+1}$ is a uniform normalized tight frame for H_N if and only if $\{f_i\}_{i=1}^{N+1}$ is unitarily equivalent to the frame $\{Pe_i\}_{i=1}^{N+1}$ where $\{e_i\}_{i=1}^{N+1}$ is an orthonormal basis for H_{N+1} and P is the orthogonal projection of H_{N+1} onto the orthogonal complement of the one-dimensional subspace of H_{N+1} spanned by $\sum_{i=1}^{N+1} e_i$.*

More general classes of uniform tight frames are the full-diversity constellations that form a group given in Ref. 17.

4 Uniform Normalized Tight Frames

There is a general method for getting finite uniform normalized tight frames:

Theorem 5. *There is a unique way to get uniform tight frames with M elements in $H = \mathbb{C}^N$ or $H = \mathbb{R}^N$. Take any orthonormal set $\{\phi_k\}_{k=1}^N$ in $\mathbb{C}^M$ which has the property*

$$\sum_{k=1}^{N}|\phi_{ki}|^2 = c, \quad \textit{for all } i.$$

Thinking of the ϕ_k as row vectors, switch to the M column vectors and divide by $\sqrt{c}$. This family is a uniform tight frame for $\mathbb{C}^N$ with M elements, and all uniform normalized tight frames for H_N with M elements are obtained in this way.

Corollary 2. *If* $\{f_i\}_{i=1}^M$ *is a uniform normalized tight frame for* H_N *then*

$$\|f_i\|^2 = \frac{M}{N}.$$

There is a detailed discussion concerning the uniform normalized tight frames for $\mathbb{R}^2$ in Ref. 11. In Ref. 17 there is a deep classification of groups of unitary operators which generate uniform normalized tight frames. The simplest case of this is the harmonic frames. In the theorem below[6], we do not assume that we have a "group" of unitaries, but instead conclude that our family of unitaries must be a group.

Theorem 6. *A family* $\{\phi_k\}_{k=0}^{M-1}$ *is a general harmonic frame for* H_N *if and only if there is a vector* $\phi_0 \in H_N$ *with* $\|\phi_0\|^2 = \frac{N}{M}$*, an orthonormal basis* $\{e_i\}_{i=1}^N$ *for* H_N *and a unitary operator* U *on* H_N *with* $Ue_i = c_i e_i$*, with* $\{c_i\}_{i=1}^N$ *distinct* M^{th}*-roots of some* $|c| = 1$ *so that* $\phi_k = U^k \phi_0$*, for all* $0 \le k \le M-1$*.*

The picture becomes much more complicated if the uniform normalzied tight frame is generated by a group of unitaries with more than one generator (see Ref. 17) or worse, if the uniform tight frame comes from a subset of the elements of such a group.

5 Frames Equivalent to Uniform Tight Frames

We saw earlier that every frame is equivalent to a normalized tight frame. That is, given any frame $\{f_i\}_{i\in I}$ with frame operator S, the frame $\{S^{-1/2}f_i\}_{i\in I}$ is a normalized tight frame. Therefore, it is natural to try to find ways to turn frames into uniform normalized tight frames. As it turns out, this is not possible in most cases. The following results come from Ref. 6:

Theorem 7. *If a frame* $\{f_i\}_{i\in I}$ *with frame operator* S *is equivalent to a uniform tight frame, then* $\{S^{-1/2}f_i\}_{i\in I}$ *is a uniform normalized tight frame. In particular, a tight frame which is not uniform cannot be equivalent to any uniform normalized tight frame.*

6 Uniform Tight Frames and Subspaces of the Hilbert Space

As we saw in Theorem 2, there is a unique way to get normalized tight frames on H_N with M-elements. Namely, we take an orthonormal basis $\{e_i\}_{i=1}^M$ for H_M and take the orthogonal projection P_{H_N} of H_M onto H_N. Then $\{P_{H_N}e_i\}_{i=1}^M$ is a normalized tight frame for H_N with M-elements. In particular, there is a natural one-to-one correspondence between the normalized tight frames for H_N with M-elements and the orthonormal bases for H_M. The uniform normalized tight frames for H_N are the ones for which $\|Pe_i\| = \|Pe_j\|$, for all $1 \le i,j \le M$. In Ref. 6 this is used to exhibit natural correspondences between these families and certains subspaces of H_M. Here we treat two frames as the same if they are unitarily equivalent.

Theorem 8. *There is a natural one to one correspondence between the normalized tight frames for H_N with M-elements and the family of all N-dimensional subspaces of H_M.*

For the uniform case we have Ref. 6:

Theorem 9. *Fix an orthonormal basis $\{e_i\}_{i=1}^M$ for H_M so that if P is the orthogonal projection of H_M onto H_N then $\{Pe_i\}_{i=1}^M$ is a uniform normalized tight frame for H_N. Then there is a natural one to one correspondence between the uniform normalized tight frames for H_N with M-elements and the subspaces W of H_M for which $\|P_W e_i\|^2 = M/N$, for all $1 \le i \le M$.*

7 Uniform Dual Frames

Another way to get uniform frames is to find them as alternate dual frames for a frame.

Definition 3. *Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space H with frame operator S. We call $\{S^{-1}f_i\}_{i\in I}$ the* **canonical dual frame** *of $\{f_i\}_{i\in I}$. In this case,*

$$f = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i, \quad \textit{for all } f \in H.$$

So the canonical dual frame can be used to reconstruct the elements of H from the frame. However, there may be other sequences in H which give reconstruction.

Definition 4. *Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space H. A family $\{g_i\}_{i\in I}$ is called an* **alternate dual frame** *for $\{f_i\}_{i\in I}$ if*

$$f = \sum_{i\in I}\langle f, g_i\rangle f_i, \quad \textit{for all } f \in H.$$

There are generally many alternate dual frames for a given frame. In fact[16], a frame has a unique alternate dual frame (i.e. the canonical dual frame) if and only if it is a Riesz basis. Moreover,[16] no two distinct alternate dual frames for a given frame are equivalent. For a normalized tight frame, its canonical dual frame is the frame itself.

We mention some results from Ref. 6 describing the existence of (uniform) tight dual frames for a given frame.

Proposition 1. *If $\{f_i\}_{i\in I}$ is a normalized tight frame for H_N, then the only normalized tight alternate dual frame for $\{f_i\}_{i\in I}$ is $\{f_i\}_{i\in I}$ itself.*

Proposition 2. *If $\{f_i\}_{i=1}^k$ is a normalized tight frame for H_N and $k < 2N$, then the only tight dual frame for $\{f_i\}_{i=1}^k$ is $\{f_i\}_{i=1}^k$ itself. If $k \ge 2N$ then there are infinitely many (non-equivalent) tight alternate dual frames for $\{f_i\}_{i=1}^k$.*

But it is really the uniform case we are interested in.

Proposition 3. *If* $\{f_i\}_{i=1}^{2N}$ *is a uniform normalized tight frame for* H_N *then there are infinitely many uniform tight alternate dual frames for* $\{f_i\}_{i=1}^{2N}$.

We are currently looking into uses for these uniform tight alternate dual frames in the setting of signal processing, as well as applications of the other results. An important goal here is to find uniform normalized tight frames with "structure" which will allow their use in signal processing. Current results show that the classes with group-based structure are quite limited. The next step is to find other classes of "computationally efficient" uniform normalized tight frames as perhaps those for which the operator matrices can be factored in some simple way.

References

[1] N.I. Akhiezer and I.M. Glazman, *Theory of linear operators in Hilbert spaces*, Frederick Ungar Publisher (1966).

[2] J.J. Benedetto and D. Colella, *Wavelet analysis of spectrogram seizure chirps*, Proc. SPIE Wavelet Appl. in Signal and Image Proc. III, Vol. 2569, San Diego, CA, July 1995, 512-521.

[3] J.J. Benedetto and G.E. Pfander, *Wavelet periodicity detection algorithms*, in Proc. SPIE Wavelet Appl. in Signal and Image Proc. VI, Vol. 3459, San Diego, CA, July 1998, 48-55.

[4] P.G. Casazza, *The Art of Frame Theory*, Taiwanese Journal of Math., Vol 4 (2) (2000) 129-202.

[5] P.G. Casazza, *Modern Tools for Weyl-Heisenberg (Gabor) frame theory*, Advances in Imaging and Electron Physics 115 (2000) 1-127.

[6] P.G. Casazza and J. Kovačević, *Uniform tight frames*, In preparation.

[7] K.L. Clarkson, W. Sweldens, and A. Zheng, *Fast multiple antenna differential decoding*, preprint.

[8] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[9] G.J. Foschini, *Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas*, Bell Labs. TEch, 1 (2) (1996) 41-59.

[10] V.K. Goyal, *Beyond traditional transform coding*, PhD thesis, Univ. California, Berkeley, 1998. Published as Univ. California Berkeley, Electron. Res. Lab. Memo. No. UCB/ERL M99/2, Jan. 1999.

[11] V.K. Goyal, J. Kovačević and J.A. Kelner *Quantized frame expansions with erasures.* ACHA, vol. 10, no. 3 (2001) 203-233.

[12] V.K. Goyal and J. Kovačević, *Optimal multiple description transform coding of Gaussian vectors*, Proc. IEEE Data Compression Conf., J.A. Storer and M. Cohn, eds., Snowbird, Utah, Mar.-Apr. 1998, 388-397.

[13] V.K. Goyal, J. Kovačević and M. Vetterli, *Multiple description transform coding: Robustness to erasures using tight frame expansions*, Proc. IEEE Int. Symp. Inform. Th., Cambridge, MA, Aug. 1998, 408.

[14] V.K. Goyal, J. Kovačević and M. Vetterli, *Quantized frame expansions as source-channel codes for erasure channels*, Proc. IEEE Data Compression Conf., J.A. Storer and M. Cohn, eds., Snowbird, Utah, Mar. 1999, 326-335.

[15] V.K. Goyal, M. Vetterli, and N.T. Thao, *Quantized overcomplete expansions in* $\mathbb{R}^N$ *: Analysis, synthesis and algorithms*, IEEE Trans. Inform. Th., (44) (1998), 16-31.16-31.

[16] D. Han and D.R. Larson, *Frames, Bases and Group Representations.* Memoirs AMS, No. 697, Providence, R.I. (2000).

[17] B. Hassibi, B. Hochwald, A. Shokrollahi, W. Sweldens, *Representation theory for high-rate multiple-antenna code design*, preprint.

[18] Heil, C. and Walnut, D.: *Continuous and discrete wavelet transforms,* SIAM Review, 31 (4) (1989) 628-666.

[19] B.M. Hochwald, T.L. Marzetta, T.J. Richardson, W. Sweldens, and R. Urbanke, *Systematic design of unitary space-time constellations*, submitted to IEEE Transactions on Information Theory.

[20] T.L. Marzetta and B.M. Hochwald, *Capacity of a mobile multiple-antenna communication line in Rayleigh flat fading*, IEEE Trans. Info. Theory 45 (1999) 139-157.

[21] R.M. Young, *An Introduction to Nonharmonic Fourier Series*, revised first edition, Academic Press (2001).