Quantized Frame Expansions in a Wireless Environment

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Abstract

We study frames for robust transmission over a multiple-antenna wireless system - BLAST. By considering as erased a component received with an SNR inferior to a given threshold, we place frames in a setting where some of the elements are deleted. In [1], the authors focused on the performance of quantized frame expansions up to $M - N$ erased components, the stucture of a frame being thus preserved. In this paper we consider every possible scenario of erasures for low-dimensionnal frames and we present optimal designs for corresponding systems using a small number of antennas.

1 Introduction

Transmission of data in a wireless environment must contend with multipath propagation, a characteristic historically viewed as an impairment which causes signal fading. Bell Labs Layered Space-Time system (BLAST) was developed to exploit this rich scattering environment, by using multiple transmitters and receivers, each with its own antenna, to create parallel subchannels that transmit independent data. This paper uses frames to provide robustness against erasures encountered in the transmission over such a wireless system. The analysis of a system with a small number of antennas and therefore with frames of low dimensions leads to optimal designs minimizing the mean-square error.

1.1 Bell Labs Layered Space Time System

BLAST is a communication technique which exploits the multipath characteristics of the wireless channel in an efficient manner to enhance capacity. Jerry Foschini and Mike Gans, worked out a theoretical framework for BLAST in 1995 [2]. Foschini

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proposed two layered space-time architectures: Diagonal BLAST (D-BLAST) and Vertical BLAST (V-BLAST), which is simpler and more practical, yet capable of attaining a hefty portion of the Shannon spectral efficiency.

BLAST assumes a rich scattering environment. It is a single user system that uses multiple transmitters and receivers, each with its own antenna, to create a number of parallel subchannels, each carrying independent data. The transmitted signals all occupy the same bandwidth simultaneously, so spectral efficiency is roughly proportional to the number of subchannels. At the receiver, BLAST uses a combination of linear and nonlinear detection techniques to disentangle the mutually interfering signals. The richer the scattering, the more subchannels can exceed the capacity obtained when the channel is treated conventionally, i.e. as a single (scalar) channel.

Transmission: We now explain V-BLAST. First a stream of data is demultiplexed into M substreams of equal rate. Each substream is encoded and associated to an antenna. The communication is organized in bursts of short enough duration so that we can assume the channel essentially unchanged during a burst. However, between two consecutive bursts, the channel may change substantially. Moreover, we assume that the channel is unknown to the transmitter but learned by the receiver through a training phase, during which known signals are transmitted and processed at the receiver. Suppose we have a rate of R bits per symbol. Then RT bits are demultiplexed in M sets, each going to one different antenna. It is as if M different users sent their message independently of one another. By "chopping up", there is no loss in overall throughput. Although the probability of error increases, the overall capacity is not reduced as the receiver knows the matrix transfer matrix H.

Processing at the Receiver: The processing at the receiver is organized around three phases: (a) interference nulling, where the interference from yet to be detected substreams is projected out, (b) interference cancelling, where interference from already detected substreams is subtracted out and (c) compensation which enables stronger elements of the received signal to compensate for the weaker elements.

Capacity Performance for Large Number of Antennas: The capacity per dimension is limited by the worst of all the M transmitted blocks, since we impose that all transmitters transmit at the same rate. Furthermore, a transmission is considered successful if all M blocks are correctly received.
By writing $M = \alpha \cdot N$, with $0 < \alpha < 1$, we get the following large-M limit:
$C_{V-BLAST} \approx max_{\{0<\alpha<1\}}\{\alpha \cdot \log_2[1 + \rho \cdot (\alpha^{-1} - 1)]\} bps/Hz/dim$,
ρ being the signal-to-noise ratio (SNR) at each receive antenna. In the limit of a large number of antennas, the V-BLAST Shannon capacity grows linearly with the number of antennas.

1.2 Frames and the Structure of the Proposed System

Frames, providing redundant representations in contrast to bases, have been used in diverse areas for different reasons. They provide resilience to additive noise [3], resilience to quantization [1], numerical stability of reconstruction [3], and greater freedom to capture significant signal characteristics [4, 5]. The redundancy of a frame can also mitigate the effect of losses in packet-based communication systems [1]. A

$$x \in \mathbb{R}^N \rightarrow \boxed{F} \xrightarrow{y\in\mathbb{R}^M} \boxed{Q} \xrightarrow{\hat{y}\in\mathcal{I}^M} \boxed{\text{BLAST transmission}} \xrightarrow{z\in\mathcal{I}^{M-e}} \boxed{\text{reconstruction}} \rightarrow \hat{x} \in \mathbb{R}^N$$

Fig. 1: Abstraction of a communication system using a quantized frame expansion. The signal vector $x \in \mathbb{R}^N$ is expanded with a frame operator F to give the frame coefficient vector $y \in \mathbb{R}^M$. The scalar quantization of y gives $\hat{y}$, which is transmitted over a BLAST system with M transmit and M receive antennas that erases some components. A reconstruction $\hat{x} \in \mathbb{R}^N$ is computed from the received vector z.

good introduction on frames can be found in [3, Ch. 3].

Similarly to [1], our aim is to exploit the resilience of frame expansions to losses. This resilience is a result of the redundancy a frame representation brings. The communication system we studied is presented Fig. 1. We denote the $\mathbb{R}^N$-valued information to be communicated by x. The source vector is represented through a frame expansion with the frame operator F, yielding $y = Fx \in \mathbb{R}^M$. The scalar quantization of the frame expansion coefficients gives $\hat{y}$ lying in a discrete subset of $\mathbb{R}^M$.

The transmission is organized around a system using BLAST with M transmit and M receive antennas. According to the SNRs coming from the M transmitters, we decide to abstract the effect of the transmission as erasures of some components of $\hat{y}$. We systematically discard a component if the SNR is below a given threshold.

The decoder receives only $M - e$ of the quantized output sequences, where e is the number of erasures during the transmission. In [1] it was assumed that there were no more than $M - N$ erasures. Now we consider every possible case by using a statistical model for the input sequence. We thus consider the source x to be a zero-mean, white, Gaussian vector with covariance matrix $R_x = \sigma^2 I_N$. We use entropy-coded uniform quantization (ECUQ). We denote the distortion-rate performance of ECUQ on a Gaussian variable with variance σ^2 by $D_{\sigma^2}(R)$. For coding at a total rate of R bits per component of x, NR bits are split among M descriptions. Thus the quantization noise power σ_η is equal to $D_{\sigma^2}(\frac{NR}{M})$. The reconstruction process is linear and uses the pseudoinverse of the frame operator. For more details about quantization and reconstruction, the reader in encouraged to consult [1].

We ultimately want to design frames that give good MSE performance, for which the first step is to compute the effect of erasures on the MSE, considering both the cases where the structure of the frame is preserved and when the number of erasures being too large, we do not have a frame anymore. We then concentrate on optimizing the frame design in systems with a small number of antennas; in particular, we study the optimization of 3×2 frames in a system with 3 transmit and 3 receive antennas.

2 Quantized Frame Expansions with Erasures

In [1], the authors studied the effect of erasures on the structure of an $M \times N$ frame and came to the conclusion that the deletion of more than $M - N$ vectors leaves something which is no more a frame but that it is possible to find uniform frames such that we still have a frame for any erasures of up to $M - N$ components. Here, we assume such frames Φ. After recalling the results found for less than $M-N$ erasures in [1], we study the scenarios with more than $M-N$ erasures, which require the definition of a statistical model for the source. We then present the criteria that will rule our optimizations.

2.1 Reconstruction with Less than $M-N$ Erasures [1]

Denote the index set of erasures by E, *i.e.* $\{\hat{y}_k\}_{k\in E}$ are lost. To the decoder it looks as if a quantized frame expansion were computed with the frame $\Phi_E = \{\varphi_k\}_{k\notin E}$, assuming Φ_E is a frame.The effective frame operator is $F_E = L_E F$, where L_E captures the losses; L_E is the $(M-e)\times M$ matrix obtained by deleting the E-numbered rows from an $M\times M$ identity matrix, with $e=|E|$.

When there are at most $M-N$ erasures, the optimal reconstruction uses the dual $\widetilde{\Phi}_E$ of the frame Φ_E, not the dual of the original frame Φ. Denote the MSE with erasure set E by MSE_E. Using the frame operator F_E associated with Φ_E, the MSE has been determined in [1]:
$MSE_E = \frac{\sigma^2}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i(F_E^*F_E)}$, where $\{\lambda_i(F_E^*F_E)\}_{i=1}^N$ is the set of eigenvalues of $F_E^*F_E$.
A useful equivalent form is $MSE_E = N^{-1}\sigma^2\mathrm{tr}((F_E^*F_E)^{-1})$.

2.1.1 MSE with One Erasure

Computing the MSE when there is one erasure from a *tight* frame gives

$$\mathrm{MSE}_1 = \left(1+\frac{1}{M-N}\right)\frac{N}{M}\sigma^2 = \left(1+\frac{1}{M-N}\right)\mathrm{MSE}_0. \quad (1)$$

This result has a simple form and is independent from the erased component. Note that deleting one element from a uniform tight frame fails to leave a frame if and only if the original frame is a basis, *i.e.* the redundancy ratio is one. The MSE (1) obtained when the original frame is tight is both average-case and minimax optimal.

2.1.2 MSE with $e\le M-N$ Erasures

Limiting our attention to tight frames, the computation of the MSE when a tight frame is subject to an arbitrary number of erasures gives

$$\mathrm{MSE}_E = \left(1+\sum_{i=1}^{e}\frac{\mu_i}{M-N\mu_i}\right)\mathrm{MSE}_0. \quad (2)$$

with μ_i being the eigenvalues of $F_E^*F_E - M/N\ I_N$. In this case, $\sum_{i=1}^{e}\mu_i = \mathrm{tr}(\varphi^*\varphi) = \mathrm{tr}(\varphi\varphi^*) = e$. The minimum of (2) is obtained when each μ_i is equal to 1—provided this is feasible.

If $e\le N$, $\mu_i=1$, $i=1,2,\ldots,e$ if and only if the erased vectors are pairwise orthogonal. Then $\mathrm{MSE}_{e \text{ orthogonal erasures}} = \left(1+\frac{e}{M-N}\right)\mathrm{MSE}_0$.

If $e>N$, it is not possible to have e eigenvalues equal to 1 because there will be at most N non-zero eigenvalues. Denoting the non-zero eigenvalues $\{\mu_i\}_{i=1}^N$,$\mathrm{MSE}_E =$

$\left(1 + \sum_{i=1}^{N} \frac{\mu_i}{M - N\mu_i}\right)$ MSE$_0$. This MSE is minimized when $\mu_i = e/N$, $i = 1, 2, \ldots, N$, which occurs when the *erased elements* form a tight frame.

2.2 MSE with $e > M - N$ Erasures

When more than $M - N$ components are erased, we do not have a frame anymore. The decoder has less than a basis representation of x. The reconstruction depends on the source which can be orthogonally decomposed as

$$x = x_S + x_S^{\perp} \text{ where } x_S \in S = span\left\{\{\phi_k\}_{k \notin E}\right\} \quad (3)$$

Since the source is Gaussian, we know that x_S and $x_S^{\perp}$ are independent. That is the reason why, the decoder not only has no direct measurement of $x^{\perp}$ but it has no way to estimate $x^{\perp}$ aside from using its mean which equals zero. The dimension of $S^{\perp}$ is $e-(M-N)$. Thus, the distortion introduced equals $\frac{1}{N}(e-(M-N))\sigma^2$. The received coefficients z provide a quantized basis representation of x_S. The basis will in general be nonorthogonal. Therefore, the distortion per component will exceed $D_{\sigma^2}(NR/M)$ by a constant factor c_e which depends on the skew of the basis. We conclude

$$\text{MSE}_E = \frac{e-(M-N)}{N}\sigma^2 + \frac{M-e}{N} c_e D_{\sigma^2}\left(\frac{NR}{M}\right). \quad (4)$$

The constant factor c_e is computed as follows: Recall that with more than $(M-N)$ erasures we do not have a frame anymore, thus $F_E^* F_E$ is not invertible. We perform a singular value decomposition of $F_E^* F_E$:

$$\mathbf{F_E^* F_E} = U \left(\begin{array}{c|c} \Lambda & 0 \\ \hline 0 & 0 \end{array}\right) U^*,$$

where U is a unitary matrix. Thus, c_e is simply given by the trace of Λ.

2.2.1 Channel Outages

Recall that we decided to abstract the transmission with BLAST by the erasure on some components. The antennas are not identical and can be dependent on one another. Thus, they have different probabilities of outage.

Example 1 *Figure 2 shows the probabilities of the different system states for a system with 4 transmit and 4 receive antennas. These probabilities have been estimated using measurements made in New York, down the Fifth Avenue. 9376 trials have been made with an SNR of 12 dB and packets of 1000 bits. According to the bit error rate, we decide to discard a packet or not.*

Let p^E be the probability of the system states, E being the set of erased components. Then, we define the overall average distortion as $\bar{D} = \sum p^E MSE_E$.

We will consider optimal the design that minimizes the overall average distortion. In the following section, we discuss optimal frame design for a case of practical interest.

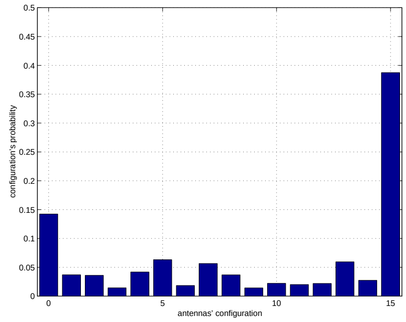


Fig. 2: Example of a probability distribution for the different system states with 4 transmit and 4 receive antennas. The value along the horizontal axis has to be converted into a binary representation (b_1 b_2 b_3 b_4), with $b_i = 0$ if substream #i fails and $b_i = 1$ otherwise.

3 Study of a Practical Case

The multiple-antenna systems with BLAST generally use a small number of antennas. We thus decided to analyze in detail the optimization in the case of a 3×2 frame, that corresponds to a system using BLAST with 3 transmit and 3 receive antennas. We consider sending a two-tuple source over three channels. The source $x = [x_1, x_2]^T$ has independent zero-mean Gaussian components with variance σ^2. The frame F produces a vector of three descriptions $y = [y_1, y_2, y_3]^T = Fx$. The descriptions are then quantized to give $\hat{y} = [\hat{y}_1, \hat{y}_2, \hat{y}_3]$. Although we discussed mostly uniform frames, here we lift the restriction and consider an arbitrary frame. Call R_y the covariance matrix of y. As the source has independent components and F is a linear operator, y is also a vector of independent components. The covariance matrix, thus diagonal, can be written as $R_y = diag(\sigma_1^2, \sigma_2^2, \sigma_3^2)$. We denote $F(i,j)$ by a_{ij}, $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

3.1 MSE with $e \leq 1$

Recall that for any number of erasures up to $M - N$ components, it is possible to find frames Φ such that Φ_E is still a frame. The dual $\widetilde{\Phi_E}$ of the frame Φ_E is used for optimal reconstruction. The MSE has been determined in Section 2: $MSE_E = N^{-1}\sigma_\eta^2 tr((F_E^* F_E)^{-1})$.

MSE with $e = 0$ In the case of no erasures, the MSE can be easily determined from the trace of $(F^*F)^{-1}$.

$$MSE_0 = \frac{1}{2}{\sigma_\eta}^2 \frac{\sum a_{ij}^2}{\sum_{i<j} (a_{i1}a_{j2} - a_{i2}a_{j1})^2}. \tag{5}$$

The distortion-rate performance of ECUQ on a Gaussian variable with variance σ^2 D_{σ^2} connects the coding rate to the quantization noise power. So we get

$${\sigma_\eta}^2 = \frac{1}{3}\left[D_{\sigma_1^2}\left(\frac{2R}{3}\right) + D_{\sigma_2^2}\left(\frac{2R}{3}\right) + D_{\sigma_3^2}\left(\frac{2R}{3}\right)\right] = \frac{1}{3}D_{\sigma^2}\left(\frac{2R}{3}\right)\left[\sum a_{ij}^2\right]. \tag{6}$$

Thus, combining (5) and (6), the expected squared error per component is given by

$$MSE_0 = \frac{1}{6} D_{\sigma^2}\left(\frac{2R}{3}\right) \frac{(\sum a_{ij}^2)^2}{\sum_{i<j}(a_{i1}a_{j2} - a_{i2}a_{j1})^2}. \tag{7}$$

MSE with $e = 1$ Now consider the case of one erased component. Let us assume that the erased component is $< x, \varphi_3 >$. $\Phi_{\{3\}}$ is still a frame and its associated frame operator is $F_{\{3\}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Using the trace of $(F_{\{3\}}^* F_{\{3\}})^{-1}$, we can write $MSE_{\{3\}}^1 = \frac{1}{2}{\sigma_\eta}^2 \frac{\sum_{i,j,i\neq 3} a_{ij}^2}{(a_{11}a_{22} - a_{12}a_{21})^2}$.
Again, D_{σ^2} connects the coding rate to the quantization noise power.

$${\sigma_\eta}^2 = \frac{1}{2}\left[D_{\sigma_1^2}\left(\frac{2R}{3}\right) + D_{\sigma_2^2}\left(\frac{2R}{3}\right)\right] = \frac{1}{3} D_{\sigma^2}\left(\frac{2R}{3}\right)\left[\sum_{i,j,i\neq 3} a_{ij}^2\right].$$

Then we have,

$$MSE_{\{3\}}^1 = D_{\sigma^2}\left(\frac{2R}{3}\right) \frac{(\sum_{i,j,i\neq 3} a_{ij}^2)^2}{4(a_{11}a_{22} - a_{12}a_{21})^2}. \tag{8}$$

Similarly, if the erased components are respectively $< x, \varphi_1 >$ and $< x, \varphi_2 >$, the mean square errors are given by :

$$MSE_{\{1\}}^1 = D_{\sigma^2}\left(\frac{2R}{3}\right) \frac{(\sum_{i,j,i\neq 1} a_{ij}^2)^2}{4(a_{21}a_{32} - a_{22}a_{31})^2} \; ; \; MSE_{\{2\}}^1 = D_{\sigma^2}\left(\frac{2R}{3}\right) \frac{(\sum_{i,j,i\neq 2} a_{ij}^2)^2}{4(a_{11}a_{32} - a_{12}a_{31})^2} \tag{9}$$

3.2 MSE with $e > 1$

MSE with $e = 2$
According to Section 2.1.2, the deletion of two components leaves a set that is no longer a frame. x can be decomposed as in (3). Suppose that the two last components were lost. The MSE has been determined in (4): $MSE_{\{2,3\}}^2 = \frac{1}{2}\sigma^2 + \frac{1}{2}c_e{\sigma_\eta}^2$. The coefficient c_e is determined using a singular value decomposition of $F_2^{\{2,3\}^*} F_2^{\{2,3\}}$:

$$F_2^{\{2,3\}^*} F_2^{\{2,3\}} = \underbrace{\frac{1}{\sqrt{a_{11}^2 + a_{12}^2}} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix}}_{U} \left(\begin{array}{c|c} \overbrace{a_{11}^2 + a_{12}^2}^{\Lambda} & 0 \\ \hline 0 & 0 \end{array}\right) \underbrace{\frac{1}{\sqrt{a_{11}^2 + a_{12}^2}} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix}}_{U^*}.$$

Finally, the coefficient c_e is given by the trace of $\Lambda^{-1} = \frac{1}{a_{11}^2 + a_{12}^2}$.
Noting that ${\sigma_\eta}^2 = D_{\sigma_1^2}\left(\frac{2R}{3}\right) = (a_{11}^2 + a_{12}^2) D_{\sigma^2}\left(\frac{2R}{3}\right)$, we get

$$MSE_{\{2,3\}}^2 = \frac{1}{2}\sigma^2 + \frac{1}{2} D_{\sigma^2}\left(\frac{2R}{3}\right) = MSE_{\{1,3\}}^2 = MSE_{\{1,2\}}^2 = MSE^2. \tag{10}$$

Note that the MSE, in the case of two erasures, does not depend on the frame. This can be explained by the fact that the source is isotropic which implies that the distortion obtained is the same whatever component is received.

MSE with $e = 3$
All the components are lost. Thus, the decoder has no way to estimate x besides using its mean. This introduces a distortion of σ^2.

3.3 Overall Average Distortion per Component

According to Section 2.2.1, the overall average distortion is

$$\begin{aligned} \bar{D} &= \underbrace{[p^{\{1,2,3\}} MSE^3 + (p^{\{1,2\}} + p^{\{1,3\}} + p^{\{2,3\}}) MSE^2]}_{\text{independent of the frame}} \\ &+ \underbrace{p^{\emptyset} MSE^0 + [p^{\{1\}} MSE_1^1 + p^{\{2\}} MSE_2^1 + p^{\{3\}} MSE_3^1]}_{\text{dependent on the frame}}. \end{aligned}$$

As the first term does not depend on the frame, we only consider the second bracketed term in the optimization. We denote this term by $\bar{D}'$. Combining (7)-(10), we can complete the computation of $\bar{D}'$.

$$\begin{aligned} \frac{\bar{D}'}{D_{\sigma^2}\left(\frac{2R}{3}\right)} &= \frac{1}{6} p^{\emptyset} \frac{(\sum a_{ij}^2)^2}{\sum_{i<j} (a_{i1}a_{j2} - a_{i2}aj1)^2} + \frac{1}{4} p^{\{1\}} \frac{(\sum_{a \neq 1} a_{ij}^2)^2}{(a_{21}a_{32} - a_{22}a_{31})^2} \\ &+ \frac{1}{4} p^{\{2\}} \frac{(\sum_{a \neq 2} a_{ij}^2)^2}{(a_{11}a_{32} - a_{12}a_{31})^2} + \frac{1}{4} p^{\{3\}} \frac{(\sum_{a \neq 3} a_{ij}^2)^2}{(a_{11}a_{22} - a_{12}a_{21})^2} \end{aligned} \tag{11}$$

We will restrict the optimization to the case of uniform frames, that is, frames with $\|\varphi_k\| = 1$ for k=1,2,3. The frame operator F can then be rewritten as $F = (\sin \alpha_k \;\; \cos \alpha_k)$ with $k = 1, 2, 3$.

Using (11), fix $\alpha_1 = 0$ and write:

$$\begin{aligned} \frac{\bar{D}'}{D_{\sigma^2}\left(\frac{2R}{3}\right)} &= \frac{3}{2} p^{\emptyset} \frac{1}{\sin^2(\alpha_2) + \sin^2(\alpha_3) + \sin^2(\alpha_3 - \alpha_2)} \\ &+ p^{\{1\}} \frac{1}{\sin^2(\alpha_3 - \alpha_2)} + p^{\{2\}} \frac{1}{\sin^2(\alpha_3)} + p^{\{3\}} \frac{1}{\sin^2(\alpha_2)}. \end{aligned} \tag{12}$$

Due to symmetry, if we suppose $(\alpha_2, \alpha_3) \in [-\pi, \pi[\times [-\pi, \pi[$ we can state that if $(\alpha_{2min}, \alpha_{3min}) \in [0, \pi/2[\times [-\pi/2, 0[$ minimizes the distortion , then $(-\alpha_{2min}, -\alpha_{3min})$, $(\alpha_{2min}, \pi + \alpha_{3min})$, $(-\alpha_{2min}, -\pi - \alpha_{3min})$,$(\alpha_{3min}, \alpha_{2min})$, $(-\alpha_{3min}, -\alpha_{2min})$,$(\pi + \alpha_{3min}, \alpha_{2min})$,$(-\pi - \alpha_{3min}, -\alpha_{2min})$ are also solutions. In addition, there is only one value of (α_2, α_3) which minimizes the distortion in $[-\pi, \pi[\times [-\pi, \pi[$.

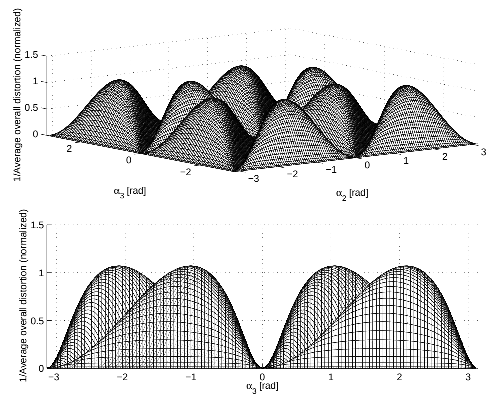


Fig. 3: $\left(\frac{\bar{D}'}{D_{\sigma^2}\left(\frac{2R}{3}\right)}\right)^{-1}$ as a function of α_2 and α_3. Eight pairs (α_2, α_3) minimize the distortion. They all correspond to uniform tight frames.

3.3.1 Case of Identical Antennas

Let us first consider the case where the antennas have equal probability of outage i.e. $p^{\{1\}} = p^{\{2\}} = p^{\{3\}} = p$. Then (12) reduces to

$$\frac{\bar{D}'}{D_{\sigma^2}\left(\frac{2R}{3}\right)} = p^{\emptyset} \frac{3}{2(\sin^2\alpha_2 + \sin^2\alpha_3 + \sin^2(\alpha_3 - \alpha_2))} + p\left[\frac{1}{\sin^2\alpha_2} + \frac{1}{\sin^2\alpha_3} + \frac{1}{\sin^2(\alpha_3 - \alpha_2)}\right].$$

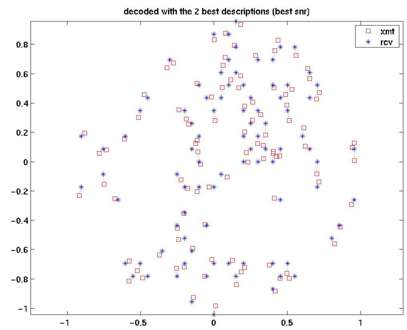
To find the frame that minimizes the distortion, we differentiate the distortion with respect to α_2 and α_3. We conclude that $\alpha_2 = k2\pi/3 + k'\pi$ and $\alpha_3 = k\pi/3$ minimizes the average overall distortion, which corresponds to the uniform tight frames. In particular $(\alpha_2 = \pi/3, \alpha_3 = -\pi/3)$ is a solution belonging to the interval $[0, \pi/2[\times[-\pi/2, 0[$.

In conclusion, a uniform tight frame minimizes the distortion and the minimum does not depend on the uniform tight frame, which confirms that there is a unique equivalence class of uniform tight frames for $N = 2$ and $M = 3$. Furthermore, the design of the frame is independent of the probabilities of failure. This comes from the fact that the uniform tight frames are optimal both in the case of zero and one erasures.

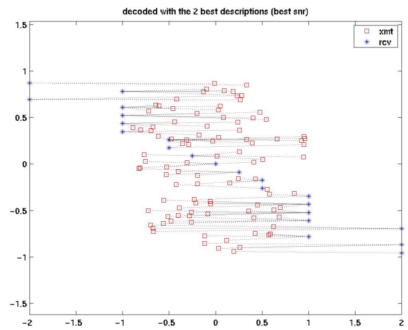
As an example, Figure 3 shows the inverse of $\frac{\bar{D}'}{D_{\sigma^2}\left(\frac{2R}{3}\right)}$ as a function of α_2 and α_3 for $p^{\emptyset} = 2/5, p^{\{1\}} = p^{\{2\}} = p^{\{3\}} = 1/6$. We can see the eight different peaks corresponding to the minima of the distortion: $(\alpha_2, \alpha_3) = (-\pi/3, -2\pi/3)$, $(\alpha_2, \alpha_3) = (-\pi/3, \pi/3)$, $(\alpha_2, \alpha_3) = (-2\pi/3, -\pi/3)$, $(\alpha_2, \alpha_3) = (-2\pi/3, 2\pi/3)$, $(\alpha_2, \alpha_3) = (\pi/3, -\pi/3)$, $(\alpha_2, \alpha_3) = (\pi/3, 2\pi/3)$, $(\alpha_2, \alpha_3) = (2\pi/3, -2\pi/3)$, $(\alpha_2, \alpha_3) = (2\pi/3, \pi/3)$.

3.3.2 Case of One Weaker Antenna

As our final analysis, let us consider the case when one antenna has a higher probability of outage. For instance, we suppose: $p^{\{3\}} \gg p^{\{1\}}, p^{\{2\}}$. We can then consider $p^{\{1\}}, p^{\{2\}} \approx 0$ as well as $p^{\emptyset} \approx 0$. Then (12) becomes $\frac{\bar{D}'}{D_{\sigma^2}\left(\frac{2R}{3}\right)} \approx p^{\{3\}} \frac{1}{\sin^2(\alpha_2)}$.
We find that the minimum distortion is reached for $\alpha_1 = 0$, $\alpha_2 = \pi/2 + k\pi$, $\alpha_3 = \alpha$,



(a) $\alpha = \pi/2$



(b) $\alpha = 3\pi/4$

Fig. 4: Example of transmitted vs. received vectors for (a) the optimal frame design and (b) a bad frame design. □ denote transmistted while $*$ denote decoded points.

for all $k \in \mathbb{Z}$. Figure 4 shows an example of transmission of vectors using respectively the optimal frame design (i.e. $\alpha = \pi/2$) and a bad frame design. The experimental setting uses one bad transmitting antenna and two good ones. The axes correspond to the coordinates of the transmitted and received vectors. We can see a remarkable difference between the two designs.

4 Conclusion

In this paper, we considered the effect of every possible case of erasures on the performance of quantized frame expansions. The optimization of the frame design in the context of a mutiple-antenna wireless system has further validated the importance of uniform tight frames. We continue to investigate the design of good frames for a larger number of antennas.

References

[1] V.K Goyal, J. Kovačević, and J.A. Kelner. Quantized frame expansions with erasures. *Journal of Appl. and Comput. Harmonic Analysis*, 10(3):203–233, May 2001.

[2] G. J. Foschini. Layered space-time architecture for wireless communication in a fading environment when using multiple antennas. *Bell Labs Technical Journal, Vol. 1, No. 2, pp 41-59*, Autumn 1996.

[3] I. Daubechies. *Ten Lectures on Wavelets.* SIAM, Philadelphia, PA, 1992.

[4] J.J. Benedetto and G.E. Pfander. Wavelet periodicity detection algorithms. In *Proc. SPIE Conf. on Wavelet Appl. in Signal and Image Proc.*, pages 48–55, San Diego, CA, July 1998.

[5] M. Unser. Texture classification and segmentation using wavelet frames. *IEEE Trans. Image Proc.*, 4:1549–1560, November 1995.