PHYSICAL LAWS GOVERNING FINITE TIGHT FRAMES

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ABSTRACT

We give a physical interpretation for finite tight frames along the lines of Columb's Law in Physics. This allows us to use results from classical mechanics to anticipate results in frame theory. As a consequence, we are able to classify those frames for an N-dimensional Hilbert space which are the closest to being tight (in the sense of minimizing potential energy) while having the norms of the frame vectors prescribed in advance. This also yields a *fundamental inequality* that all finite tight frames must satisfy.

1. INTRODUCTION

If $\mathbb{H}$ is a Hilbert space, a sequence $\{\varphi_n\}_{n=1}^M$ (M is finite or infinite) is a *frame* for $\mathbb{H}$ if there are constants $A, B > 0$ so that for all $\varphi \in \mathbb{H}$,

$$A\|\varphi\|^2 \leq \sum_{n=1}^{M} |\langle \varphi, \varphi_n\rangle|^2 \leq B\|\varphi\|^2.$$

If $A = B = \lambda$, $\{\varphi_n\}_{n=1}^M$ is a λ-*tight frame*. If $\lambda = 1$, it is a *Parseval frame*; if $\|\varphi_n\| = \|\varphi_m\|$ for all $1 \leq n, m \leq M$ it is a *equal-norm* frame; and if $\|\varphi_n\| = 1$ for all n it is a *unit-norm* frame. The importance of λ-tight frames is that they allow simple *reconstruction* of the elements of $\mathbb{H}$. It is known that $\{\varphi_n\}_{n=1}^M$ is a frame for $\mathbb{H}$ if and only if

$$S\varphi = \sum_{n=1}^{M} \langle \varphi, \varphi_n\rangle \varphi_n,$$

is an invertible operator on $\mathbb{H}$ called the *frame operator*. To reconstruct an element $\varphi \in \mathbb{H}$ we write

$$\varphi = SS^{-1}\varphi = \sum_{n=1}^{M} \langle S^{-1}\varphi, \varphi_n\rangle \varphi_n.$$

So reconstruction requires inverting the frame operator which is often difficult or impossible in practice. It follows that for all $\varphi \in \mathbb{H}$ we have

$$\langle A\varphi, \varphi\rangle = A\|\varphi\|^2 \leq \sum_{n=1}^{M} |\langle \varphi, \varphi_n\rangle|^2 = \langle S\varphi, \varphi\rangle \leq B\|\varphi\|^2 = \langle B\varphi, \varphi\rangle.$$

Hence, $AI \leq S \leq BI$ and so our frame is λ-tight if and only if $S = \lambda I$. So if $\{\varphi_n\}_{n=1}^M$ is a λ-tight frame then for all $\varphi \in \mathbb{H}_N$,

$$\varphi = \frac{1}{\lambda} \sum_{n=1}^{M} \langle \varphi, \varphi_n\rangle \varphi_n.$$

So for applications we need to construct tight frames so the frame operator is immediately invertable.

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Thanks: P.G.C. and M.T.L. were supported by NSF DMS 0102686

2. COLUMB'S LAW AND THE FRAME FORCE

The authors of[2] investigated the common notions of what it means to equally distribute a collection of electrons upon a conductive spherical shell. Physically, in the absence of external forces the charged particles will repel each other according to the inverse-square Columb force law

$$CF(\phi_m, \phi_n) = \frac{\phi_m - \phi_n}{\|\phi_m - \phi_n\|^3}.$$

Intuitively, the corresponding optimal arrangements are those which minimize the internal pressure of the points upon each other. Specifically, given M distinct electrons located at points $\{\phi_m\}_{m=1}^M$, one seeks to minimize the total corresponding potential energy of the system,

$$CP(\{\phi_m\}_{m=1}^M) = \sum_{m=1}^{M} \sum_{n \neq m} \frac{1}{\|\phi_m - \phi_n\|}.$$

And though only a global minimizer corresponds to true equidistribution in this context, local minimizers are also of physical interest, in that they correspond to collections of points in equilibrium.

It is known that local minimizers of potential energy need not be global minimizers. For example, a dodecahedron inscribed in a sphere is a local minimizer of potential energy while there are different configurations with smaller total potential energy.

In,[1] Benedetto and Fickus defined a central force between real unit norm vectors ϕ_m and ϕ_n (called the *frame force*) by

$$FF(\phi_m, \phi_n) = 2\langle \phi_m, \phi_n \rangle (\phi_m - \phi_n).$$

The important point here is that the frame force for orthogonal vectors is 0. Also, vectors having an acute angle between them are repelling while vectors having an obtuse angle are attracting. So "charged particles" under the frame force are trying to reach equilibrium by becoming as orthogonal as possible. There are some other unnatural aspects to the frame force. For one, it needs a universal reference point—a fixed origin. Also, the force field generated by a point is not conservative. That is, the work required to get from one point to another depends upon the particular path taken. However (and this is all we need for our applications) the frame force is conservative when the points are constricted to lie on a sphere.

3. THE FRAME POTENTIAL

Since we need our vectors to lie on possibly different spheres, we need a corresponding *weighted frame force (potential).* This makes the physical systems much more difficult to understand intuitively. That is, visualizing the movements of M charged particles restricted to M concentric spheres can challenge the imagination. However, this can be greatly simplified by "projecting" the dynamics down onto the unit sphere.

Consider two points, each of whose movement is restricted to a sphere of a given, yet arbitrary radius. That is, given $a_m, a_n > 0$, consider ϕ_m with $\|\phi_m\| = a_m$ and ϕ_n with $\|\phi_n\| = a_n$. Note that

$$\begin{aligned} \|\phi_m - \phi_n\|^2 &= \|\phi_m\|^2 - 2\langle \phi_m, \phi_n \rangle + \|\phi_n\|^2, \\ &= a_m^2 - 2\langle \phi_m, \phi_n \rangle + a_n^2, \end{aligned}$$

and so we may rewrite the frame force between these points as

$$\begin{aligned} FF(\phi_m, \phi_n) &= 2\langle \phi_m, \phi_n \rangle (\phi_m - \phi_n), \\ &= (a_m^2 + a_n^2 - \|\phi_m - \phi_n\|^2)(\phi_m - \phi_n). \end{aligned}$$

As in the Columb case, the pairwise potential between these points may be found by integrating the "magnitude" of this central force,

$$p(x) = -\int (a_m^2 + a_n^2 - x^2) x\, dx = \frac{1}{4} x^2 [x^2 - 2(a_m^2 + a_n^2)],$$

and evaluating at $x = \|\phi_m - \phi_n\|$,

$$P(\phi_m, \phi_n) = p(\|\phi_m - \phi_n\|) = \langle \phi_m, \phi_n \rangle^2 - \frac{1}{4}(a_m^2 + a_n^2)^2.$$

The total potential contained within the physical system is the sum of all pairwise potentials,

$$TP(\{\phi_m\}_{m=1}^M) = \sum_{m,n} |\langle \phi_m, \phi_n \rangle|^2 - \frac{1}{4}\sum_{m,n}(a_m^2 + a_n^2)^2$$

However, we may disregard the additive constant, as it has no physical significance. This then leads to the *frame potential* definition of Benedetto and Fickus.[1]

Definition 3.1. The frame potential of a sequence $\{\phi_m\}_{m=1}^M \subseteq \mathbb{H}_N$ is

$$FP(\{\phi_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{n=1}^M |\langle \phi_m, \phi_n \rangle|^2.$$

Remark 3.2. Given two sequences $\{\phi_m\}_{m=1}^M$ and $\{\psi_m\}_{m=1}^M$ in $\mathbb{R}^N$ with $\|\phi_m\| = a_m = \|\psi_m\|$ for all m, $FP(\{\psi_m\}_{m=1}^M) - FP(\{\phi_m\}_{m=1}^M)$ is the work required to transform $\{\phi_m\}_{m=1}^M$ into $\{\psi_m\}_{m=1}^M$ while remaining on the spheres of radii $\{a_m\}_{m=1}^M$.

The frame potential is measuring how close a frame is to being orthogonal.

For any frame $\{\varphi_n\}_{n=1}^M$ for $\mathbb{H}_N$ with frame bounds A, B, and frame operator S we have

$$A\|\varphi_n\|^2 \leq \sum_{m=1}^M |\langle \varphi_n, \varphi_m \rangle|^2 \leq B\|\varphi_n\|^2.$$

Hence,

$$A\sum_{n=1}^M \|\varphi_n\|^2 \leq FP(\{\varphi_n\}_{n=1}^M) \leq B\sum_{n=1}^M \|\varphi\|^2.$$

It is known that

$$\text{Trace}S = \sum_{n=1}^M \|\varphi_n\|^2.$$

In particular:

$$A\text{Trace}S \leq FP(\{\varphi_n\}_{n=1}^M) \leq B\text{Trace}S.$$

For a tight frame, $S = AI$ so $\text{Trace}S = NA$ and

$$FP(\{\varphi_n\}_{n=1}^M) = NA^2.$$

So for a Parseval frame (and hence for an orthonormal basis)

$$FP(\{\varphi_n\}_{n=1}^M) = N.$$

4. MINIMIZERS OF THE FRAME POTENTIAL

The frame potential is measuring how close a frame is to being orthogonal. In particular, we will see that if $\mathcal{F}$ is the family of frames with lower frame bound λ then the λ-tight frames are the minimizers of the frame potential over $\mathcal{F}$. This theorem gives us a way to identify tight frames. i.e. They are the minimizers of the frame potential on certain families of frames. Our goal is to identify those families of frames which have minimizers of the frame potential and for which these minimizers must be tight.

The important point here is the fact that the minimizers of the frame potential are tight frames[1] (the proof below is new).

Proposition 4.1. Let $\mathbb{H}_N$ be an N-dimensional Hilbert space, $0 < \lambda$ and let

$$W = \left\{ \{\phi_m\}_{m=1}^M | \sum_{m=1}^M \|\phi_m\|^2 = \lambda \right\}.$$

1. If $M \le N$, the minimum value of the frame potential on W is λ^2/M and the minimizers are orthogonal sequences of vectors all with the same norm $\sqrt{\lambda/M}$.
2. If $M \ge N$, the minimum value of the frame potential on W is λ^2/N and the minimizers are the tight frames with tight frame bound λ/N.

Proof.

1. We compute:

$$\begin{aligned} FP(\{\phi_m\}_{m=1}^M) &= \sum_{n,m=1}^m |\langle \phi_n, \phi_m \rangle|^2 \\ &= \sum_{m=1}^M \|\phi_m\|^4 + \sum_{n\neq m} |\langle \phi_n, \phi_m \rangle| \\ &\ge \sum_{m=1}^M \|\phi_m\|^4. \end{aligned}$$

Since we assumed that $\sum_{m=1}^M \|\phi_m\|^2 = \lambda$, by a standard application of Lagrange multipliers, the right-hand side of (1) is minimized when

$$\|\phi_m\|^2 = \|\phi_n\|^2, \quad \text{for all } 1 \le m, n \le M.$$

In this case $\|\phi_m\|^2 = \lambda/M$, for all $1 \le m \le M$ and so

$$\sum_{m=1}^M \|\phi_m\|^4 = \frac{\lambda^2}{M}.$$

This minimum is achieved when we have equality in (1), and hence

$$\sum_{n\neq m} |\langle \phi_n, \phi_m \rangle|^2 = 0,$$

showing that $\{\phi_m\}_{m=1}^M$ is an orthogonal sequence.

2. When $M > N$, we cannot use the same approach, since we cannot find M mutually orthogonal vectors ϕ_m. By (1), minimizing the frame potential under our constraint means minimizing $\sum_{n=1}^n \lambda_n^2$ under the constraint $\sum_{n=1}^N \lambda_n = \sum_{m=1}^M \|\phi_m\|^2 = \lambda$. Again, a standard application of Lagrange multipliers yields that the minimizers satisfy $\lambda_n = \lambda/N$, for all $1 \le n \le N$. Hence, the frame operator for $\{\phi_m\}_{m=1}^M$ is $(\lambda/N)I$. That is, a minimizer of the frame potential is a tight frame with the tight frame bound λ/N. Since there always exist such tight frames and these are clearly minimizers of the frame potential, we have the proof.

□

We have immediately:

Corollary 4.2. Given $\{\phi_m\}_{m=1}^M \subseteq H_N$,

$$FP(\{\phi_m\}) = \sum_{m=1}^M \sum_{n=1}^M |\langle \phi_n, \phi_m\rangle|^2 \geq \frac{\left(\sum_{m=1}^M \|\phi_m\|^2\right)^2}{N}$$

with equality if and only if $\{\phi_m\}_{m=1}^M$ is a tight frame.

An important result of Benedetto and Fickus[1] is that local minimizers of the frame potential for unit norm frames are also global minimizers. We present here a generalization of this.

Proposition 4.3. Let $\{\varphi_m\}_{m=1}^M$ be a frame for $\mathbb{H}_N$ with frame operator S and eigenspaces $\{E_{\lambda_i}\}_{i=1}^L$ and $\lambda_1 > \lambda_2 > \cdots \lambda_L$. Define

$$F : S^{\mathbb{K}^N} \to \mathbb{R},$$

(where K is either $\mathbb{R}$ or $\mathbb{C}$) by

$$F(\Phi) = \sum_{m=1}^M |\langle \Phi, \phi_m\rangle|^2.$$

Then the sphere of E_{λ_L} is the set of local minimizers of F. Hence, the local minimizers of F are global minimizers.

Proof. We will do the real case since it is quite illuminating. The complex case follows from the same proof with notational changes. Since F is continuous and non zero and $\{\varphi_m\}_{m=1}^M$ spans $\mathbb{K}^N$, by Lagrange Multipliers, there is a $\lambda \neq 0$ so that the minimizers of F satisfy:

$$\nabla F = \lambda \nabla G,$$

where for $\Phi = (\Phi_1, \Phi_2 \cdots, \cdots, \Phi_n)$ we have

$$G(\Phi) = \sum_{i=1}^N \Phi_i^2 - 1.$$

Hence,

$$\partial_i \lambda G = 2\lambda \Phi_i, \quad \text{for all } 1 \leq i \leq N.$$

Also,

$$F(\Phi) = \sum_{m=1}^M \left[\sum_{k=1}^N \Phi_k \langle \varphi_m, e_k\rangle\right]^2.$$

Hence,

$$\partial_i F = \sum_{m=1}^M 2 \left[\sum_{k=1}^N \Phi_k \langle \varphi_m, e_k\rangle\right] \langle \varphi_m, e_i\rangle = 2 \sum_{m=1}^M \langle \Phi, \varphi_m\rangle \langle \varphi_m, e_i\rangle.$$

Finally,

$$\Phi_i = (2\lambda)^{-1} \sum_{m=1}^M \langle \Phi, \varphi_m\rangle\langle \varphi_m, e_i\rangle = (2\lambda)^{-1} \left\langle \left(\sum_{m=1}^M \langle \Phi, \varphi_m\rangle \varphi_m\right), e_i \right\rangle.$$

Hence,

$$\Phi = (2\lambda)^{-1} \sum_{m=1}^M \langle \Phi, \varphi_m\rangle \varphi_m.$$

We now have that a minimizer of F is an eigenvector of S. Next we show that these eigenvectors are all in E_{λ_L}. We proceed by way of contradiction. If Φ is a local minimizer of F and $\Phi \in E_{\lambda_i}$ for some $1 \leq i < L$, choose $e_1 \in E_{\lambda_L}$ with $\|e_1\| = 1$. Fix $0 < \epsilon < 1$ and let

$$\Psi = \epsilon \Phi + \sqrt{(1-\epsilon^2)} e_1.$$

So $\|\Psi\| = 1$ and

$$F(\psi) = \sum_{m=1}^{M} |\langle \Psi, \varphi_m\rangle|^2 = \epsilon^2\lambda_i + (1-\epsilon^2)\lambda_L < \lambda_i = F(\Phi).$$

This contradiction completes the proof. □

Definition 4.4. A frame $\{\varphi_m\}_{m=1}^M$ for $\mathbb{H}_N$ with frame operator S is called a *FF-critical sequence* if each ϕ_m is an eigenvector for S.

We now have,

Proposition 4.5. If $\{\varphi_m\}_{m=1}^M$, $M \geq N$ is a sequence of vectors in $\mathbb{H}_N$ with frame operator S over its span, and ϕ_i is a local minimizer for the frame potential over the set:

$$\Omega = \{\{\varphi_m\}_{m\neq i} \cup \{\Phi\} \mid \Phi \in \mathbb{H}_N \text{ and } \|\Phi\| = \|\varphi_i\|\},$$

then φ_i is an eigenvector for S. Hence, any locally minimal sequence for the frame potential over Ω is an FF-critical sequence. Moreover, any minimizer for the frame potential over Ω must span $\mathbb{H}_N$.

Proof. An obvious compactness argument guarantees that there is some sequence $\{\varphi_m\}_{m=1}^M \in \Omega$ which minimizes the frame potential. It is not clear, and will be addressed at the end of the proof, that the minimizers span the space $\mathbb{H}_N$. We note that:

$$\begin{aligned} FP(\{\{\varphi_m\}_{m\neq i} \cup \{\Phi\}) &= \sum_{n,m\neq i} |\langle \varphi_n, \varphi_m\rangle|^2 + |\langle \Phi, \Phi\rangle|^2 + 2\sum_{m\neq i} |\langle \Phi, \varphi_n\rangle|^2 \\ &= FP(\{\varphi_m\}_{m\neq i}) + 2F(\{\Phi\}) - \|\varphi_i\|^4, \end{aligned}$$

where

$$F(\{\Phi\}) = \sum_{m\neq i} |\langle \Phi, \varphi_m\rangle|^2.$$

It follows that the minimizers of FP over Ω are the minimizers of F. By Proposition 4.3, φ_i is an eigenvector for $\{\varphi_m\}_{m\neq i}$ with eigenvalue say λ. Now,

$$\begin{aligned} \sum_{m=1}^{M} \langle \varphi_i, \varphi_m\rangle \varphi_m &= \sum_{m\neq i} \langle \varphi_i, \varphi_m\rangle \varphi_m + \langle \varphi_i, \varphi_i\rangle \varphi_i \\ &= \lambda\varphi_i + \|\varphi_i\|^2\varphi_i = (\lambda + \|\varphi_i\|^2)\varphi_i. \end{aligned}$$

Suppose $\{\varphi_m\}_{m=1}^M \in \Omega$ is a minimizer for the frame potential over Ω. We proceed by way of contradiction. Suppose span $\{\varphi_m\}_{m=1}^M \neq \mathbb{H}_N$. Let S be the frame operator for $\{\varphi_m\}_{m=1}^M$ over its span with eigenspaces $\{E_{\lambda_i}\}_{i=1}^L$. By the first part of the proof, the vectors ϕ_m sit in the E_{λ_i}. For every $1 \leq j \leq L$ let $I_j = \{m \mid \phi_m \in E_{\lambda_i}\}$. If $\{\psi_m\}_{m=1}^M$ is a λ-tight frame, it is known (and a simple calculation to verify) that $\|\psi_i\|^2 \leq \lambda$ and $\|\psi_i\|^2 = \lambda$ if and only if $\psi_i \perp$ span $\{\psi_m\}_{m\neq i}$. Now, since $M \geq N$ and span $\{\varphi_m\}_{m=1}^M \neq H_N$, it follows that there is a $1 \leq j \leq L$ so that $|I_j| > \dim E_{\lambda_j}$. Hence, there is some $i \in I_j$ with $\|\varphi_i\|^2 < \lambda_j$. Now choose $e_1 \in \mathbb{H}_N$ with $\|e_1\| = 1$ and $e_1 \perp \{\varphi_m\}_{m=1}^M$. Define $\{\psi_m\}_{m=1}^M \in \Omega$ by: $\psi_m = \phi_m$ for all $1 \leq m \neq i \leq M$ and let $\psi_i = \|\varphi_i\| e_1$. We will obtain a contradiction by showing that $FP(\{\psi_m\}_{m=1}^M) < FP(\{\varphi_m\}_{m=1}^M)$.

$$\begin{aligned} FP(\{\psi_m\}_{m=1}^M) &= FP(\{\psi_m\}_{m\neq i} + \|\psi_i\|^4 \\ &= FP(\{\varphi_m\}_{m\neq i}) + \|\varphi_i\|^4 \\ &= \sum_{n,m\neq i} |\langle \varphi_n, \varphi_m\rangle|^2 + \|\varphi_i\|^4 \\ &= \sum_{n,m=1}^{M} |\langle \varphi_n, \varphi_m\rangle|^2 + \|\varphi_i\|^4 - 2\sum_{n=1}^{M} |\langle \varphi_i, \varphi_n\rangle|^2 + \|\varphi_i\|^4 \\ &= FP(\{\varphi_m\}_{m=1}^M) + 2\|\varphi_i\|^4 - 2\lambda_j\|\varphi_i\|^2 \\ &= FP(\{\varphi_m\}_{m=1}^M) + 2\|\varphi_i\|^2(\|\varphi_i\|^2 - \lambda_j) < FP(\{\varphi_m\}_{m=1}^M). \end{aligned}$$

This contradiction completes the proof of the Theorem. □

The previous proposition gives a fairly exact form for local minimizers of the frame potential.

5. THE WEIGHTED FRAME FORCE (POTENTIAL)

Though the frame potential is the potential energy contained within a system of points of equal weight on spheres of varying radii, we form an equivalent situation by considering points of varying weight all lying on a common sphere.

Definition 5.1. Let S denote the unit sphere in $\mathbb{R}^N$.

- The *weighted frame force* is

$$\begin{aligned} WFF : [0,\infty) &\times [0,\infty)\times S\times S \to \mathbb{R}^N, \\ WFF(w_m, w_n, \psi_m, \psi_n) &= 2w_m w_n \langle \psi_m, \psi_n\rangle (\psi_m - \psi_n). \end{aligned}$$

- The *weighted frame potential* is

$$WFP(\{w_m\}_{m=1}^M, \{\psi_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{n=1}^M w_m w_n |\langle \psi_m, \psi_n\rangle|^2.$$

We write,

$$WFP(\{a_m^2\}_{m=1}^M, \{\psi_m\}_{m=1}^M) = FP(\{\phi_m\}_{m=1}^M)$$

Thus, we shall no longer view the frame potential as the total potential energy of a system of points of mass 1 restricted to spheres of radius a_m. Rather, it shall be perceived more naturally as the energy of a system of points of masses a_m^2 restricted to a single sphere of radius 1.

In our situation, the points experiencing the frame force are constrained to move upon spheres. Therefore, only the components of the frame force acting upon a point which are tangent to the surface of the sphere at that point make a contribution to the frame potential. Thus, given $\phi_m, \phi_n \in \mathbb{R}^N$ with $\|\phi_m\| = a_m$, we wish to explicitly find the component of $FF(\phi_m, \phi_n)$ which lies tangent to the sphere of radius a_m at ϕ_m. Of course, to find the tangential component, we need only subtract the normal component from the whole. And since the surface in question is a sphere, the normal component of the force is simply the projection of the force onto the line passing through ϕ_m, i.e.

$$\frac{\langle FF(\phi_m, \phi_n), \phi_m\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m.$$

We therefore simplify

$$\begin{aligned} &FF(\phi_m, \phi_n) - \frac{\langle FF(\phi_m, \phi_n), \phi_m\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m \\ &\quad= \langle \phi_m, \phi_n\rangle(\phi_m - \phi_n) - \langle \phi_m, \phi_n\rangle \frac{\langle \phi_m, \phi_m\rangle - \langle \phi_m, \phi_n\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m \\ &\quad= \langle \phi_m, \phi_n\rangle \left(\frac{\langle \phi_m, \phi_n\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m - \phi_n \right). \end{aligned}$$

This leads us to:

Proposition 5.2. Given a sequence $\{\phi_m\}_{m=1}^M \subseteq \mathbb{R}^N$ with $\|\phi_m\| = a_m$, the following are equivalent

1. $\{\phi_m\}_{m=1}^M$ is a tight frame for $\mathbb{R}^N$,
2. $\sum_{m=1}^M FF(\phi, \phi_m) = 0$ for $\phi \in \mathbb{R}^N$,
3. $\sum_{m=1}^M FF(w, a_m^2, \phi, \phi_m) = 0$ for all $\phi \in S^{(N-1)}$.

6. MINIMIZERS OF THE WEIGHTED FRAME POTENTIAL

To identify the minimizers of the weighted frame potential, we need an observation about decreasing sequences of positive numbers. Basically, this result compares the terms of the sequence to the *dimensional average* of the following terms.

Proposition 6.1. Given any sequence $\{a_m\}_{m=1}^M \subset \mathbb{R}$ with $a_1 \geq \cdots \geq a_M \geq 0$, and any $N \leq M$, there is a unique index N_0 with $1 \leq N_0 \leq N$, such that the inequality

$$a_n > \frac{\sum_{m=n+1}^M a_m}{N-n} \tag{1}$$

holds for $1 \leq m < N_0$, while the opposite inequality

$$a_n \leq \frac{\sum_{m=n+1}^M a_m}{N-n} \tag{2}$$

holds for $N_0 \leq m \leq N$.

Proof. Let $I = \{n : (N-n)a_n \leq \sum_{m=n+1}^M a_m\}$. Now $N \in I \neq \phi$. Also, if $n \in I$ then $n+1 \in I$ as the following shows:

$$\begin{aligned}
[N-(n+1)]a_{n+1} &= -a_{n+1} + (N-n)a_{n+1} \\
&\leq -a_{n+1} + (N-n)a_n \\
&\leq -a_{n+1} + \sum_{m=n+1}^M a_M \\
&= \sum_{m=n+2}^M a_m
\end{aligned}$$

Letting N_0 be the minimum index in I completes the proof. □

Corollary 6.2. Let $a_1 \leq a_2 \geq \cdots \geq a_M > 0$ and $N \leq M$. The following are equivalent:

1. For all $1 \leq d < N$,
$$a_d^2 \leq \frac{\sum_{m=d+1}^M a_m^2}{N-d}$$
2. $\sum_{m=1}^M a_n^2 \geq N a_1^2$
3. If $\lambda = \sqrt{\frac{N}{\sum_{m=1}^M a_m^2}}$,then $\lambda a_m \leq 1$, for all $1 \leq m \leq M$.

Moreover, if $\{\phi_m\}_{m=1}^M$ is a tight frame for H_N with $\|\phi_1\| \geq \|\phi_2\| \geq \cdots \geq \|\phi_M\|$, then $\{\|\phi_m\|^2\}_{m=1}^M$ satisfies 1-3 above.

Proof. For 1 implies 2, let $d = 1$ in 1. 2 implies 1 follows from Proposition 4.1 since 2 implies that $N_0 = 1$. The equivalence of 1 and 3 is immediate.

For the *moreover* part, let $\{e_n\}_{n=1}^M$ be an orthonormal basis for H_N with $\phi_1 = a_1 e_1$. Then

$$\|\phi_1\|^2 \leq \sum_{m=1}^M |\langle \phi_m, e_1\rangle|^2 = \frac{\text{Tr}S}{N} = \frac{\sum_{m=1}^M \|\phi_m\|^2}{N}$$

□

We call 1 in the above proposition the *fundamental inequality for tight frames.*

The proof of the next theorem is a serious piece of work relying heavily on a deep intuitive understanding of the frame potential, and so we refer the reader to[2] for the details.

We will denote by $S(a_1, \cdots, a_m)$ the product of spheres in $\mathbb{R}^N$ of radii $\{a_1, \cdots, a_m\}$.

Theorem 6.3. Given a sequence $a_1 \geq a_2 \geq \cdots \geq a_M > 0$ and any $N \leq M$, let N_0 be the smallest index n for which

$$a_n^2 \leq \frac{\sum_{m=n+1}^{M} a_m^2}{N-n}$$

holds. Then any local minimizer of the frame potential

$$FP : S(a_1, \cdots, a_m) \to \mathbb{R}$$

is of the form

$$\{\phi_m\}_{m=1}^{M} = \{\phi_m\}_{m=1}^{N_0-1} \cup \{\phi_m\}_{m=N_0}^{M}$$

where $\{\phi_m\}_{m=1}^{N_0=1}$ is an orthogonal set for whose orthogonal complement $\{\phi_m\}_{m=N_0}^{M}$ forms a tight frame.

An important consequence of Theorem 6.3 is that we can only get tight frames for H_N when $N_0 = 1$. This case can be summarized as:

Corollary 6.4. Fix $N \leq M$ and $a_1 \geq a_2 \geq \cdots \geq a_M > 0$. The following are equivalent:

1. There is tight frame $\{\phi_m\}_{m=1}^{M}$ for H_n satisfying $\|\phi_m\| = a_m$, for all $1 \leq m \leq M$.
2. $\{a_m\}_{m=1}^{M}$ satisfies the fundamental inequality for tight frames.

As we discussed earlier, local minimizers of the Columb potential are not necessarily global minimizers. However, for the frame potential, we do have that local minimizers are global minimizers. Hence, the characterization of Theorem 6.3 applies to all minimizers and yields the following result from.[2]

Corollary 6.5. Given a sequence $\{a_m\}_{m=1}^{M} \subset \mathbb{R}$ with $a_1 \geq \cdots \geq a_M > 0$, and any $N \leq M$, let N_0 denote the smallest index n such that

$$a_n^2 \leq \frac{\sum_{m=n+1}^{M} a_n^2}{N-n}$$

holds. Then, for the frame potential $FP : S(a_1, \cdots, a_M) \to \mathbb{R}$,

1. The minimal value is $\sum_{m=1}^{N_0-1} a_m^4 + \frac{(\sum_{m=N_0}^{M} a_m^2)^2}{N-N_0+1}$.
2. Any local minimizer is a global minimizer.
3. The minimizers are precisely those sequences where $\{\phi_m\}_{m=1}^{N_0-1}$ is an orthogonal set for whose orthogonal complement $\{\phi_m\}_{m=N_0}^{M}$ forms a tight frame.

7. CONCLUDING REMARKS

Finite tight frames are fundamental for a broad spectrum of frame applications. Until now it was thought that such frames were sparse and they could not be constructed for many applications. This paper,[2] combined with a considerable number of recent refinements[3–8] shows that actually tight frames are everywhere and can be *custom built* for most applications.

REFERENCES

1. J.J. Benedeto and M. Fickus, *Frame Potentials*, Advances in Computational Math, **vol. 18** Nos. 2-4 (2003), 357-385.
2. P.G. Casazza, M. Fickus, J. Kovačević, M. Leon and J. Tremain, *A physical interpretation for tight frames*, preprint.
3. P.G. Casazza, M. Fickus, J. Kovačević, M. Leon and J. Tremain, *Representations of frames*, preprint.
4. P.G. Casazza, M. Fickus, J. Kovačević, M. Leon, and J. Tremain, *Constructing infinite tight frames*, preprint.
5. P.G. Casazza and J. Kovačević, *Equal norm tight frames with erasures*, Advances in Computation Math., **vol. 18** Nos. 2-4 (2003) 387-430.
6. P.G. Casazza and M. Leon, *Existence and construction of finite tight frames*, preprint.
7. P.G. Casazza and M. Leon, *Frames with a given frame operator*, preprint.
8. P.G. Casazza and M. Leon, *Frameware*, 2002.