

A New Class of Real Lapped Tight Frame Transforms Derived by Seeding

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ABSTRACT

We propose a design procedure for the real, equal-norm, lapped tight frame transforms (LTFTs). These transforms have been recently proposed as both a redundant counterpart to lapped orthogonal transforms and an infinite-dimensional counterpart to harmonic tight frames. In addition, LTFTs can be efficiently implemented with filter banks. The procedure consists of two steps. First, we construct new lapped orthogonal transforms designed from submatrices of the DFT matrix. Then we specify the seeding procedure that yields real equal-norm LTFTs. Among them we identify the subclass of maximally robust to erasures LTFTs.

Keywords: Frames, tight, bases, filter bank, lapped transform, DFT, paraunitary

1. INTRODUCTION

Redundancy has become a common tool in signal processing and communications in recent years. It found its way into signal representations through frames,¹⁻³ which serve a wide range of applications from robust transmission to denoising⁴ to the classification of different biomedical image datasets.⁵⁻⁷ In this paper, we design new classes of frames that can be tailored to the various applications that require redundancy.

We call a redundant set of vectors $\{\varphi_i\}$, $i \in \mathbb{Z}$, that span $\ell^2(\mathbb{Z})$, a *frame*. A signal $x \in \ell^2(\mathbb{Z})$ is expanded into the frame using a *transform*, which computes the signal projection coefficients. The original signal then is reconstructed using the corresponding inverse transform

$$x = \Phi X = \underbrace{\Phi \tilde{\Phi}^* x}_{\text{transform}} \quad . \quad (1)$$

$(\cdot)^*$ denotes the Hermitian transpose. The columns of Φ are the frame vectors φ_i . In this paper, we view both Φ and its *dual* $\tilde{\Phi}$ as infinite matrices.

In this paper, we design what we call *lapped tight frame transforms (LTFTs)*, which are: 1) *lapped*: the support of each φ_i is longer than a single signal block processed by the filter bank; 2) *tight*: $\tilde{\Phi} = \Phi$, and signal reconstruction is performed as $\Phi\Phi^* = I$; 3) *equal-norm*: $\|\varphi_i\| = \|\varphi_j\|$ for any $i, j \in \mathbb{Z}$; 4) *maximally robust to erasures*: a signal can still be reconstructed after partial data loss. In addition, such transforms can be implemented with real filter banks, which make them very efficient.

The above requirements resemble those of *lapped orthogonal transforms (LOTs)*.⁸ LOTs are expansions into orthonormal bases that are computationally efficient and can be implemented with filter banks. In our previous work,⁹ we constructed LTFTs as submatrices of LOTs, using a process called *seeding*. In this paper, we systematically construct a large class of new real LOTs and then seed them to obtain a new class of real LTFTs.

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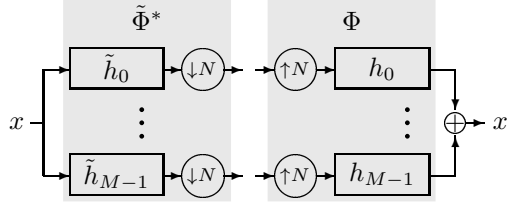


Figure 1. An M -channel filter bank sampled by N implementing a basis ($M = N$) or a frame ($M > N$) expansion. The analysis part computes the transform $X = \tilde{\Phi}^* x$; and the synthesis part computes the inverse transform $x = \Phi X$.

Related work includes the construction of a transform derived from the extended lapped complex transform.^{10,11} The authors' approach is similar to ours, but it does not use seeding and yields a completely different LTFT.¹² The same authors have also developed a two-dimensional nonseparable LTFT.¹³

2. BACKGROUND

In this section we discuss signal transforms that can be implemented with multichannel filter banks. Such transforms can be interpreted as expansions into bases or frames that are implemented with critically-sampled or oversampled filter banks, respectively. We focus on basis and frame vectors with overlapping support to avoid blocking effects. We also describe the seeding process, and discuss the construction of tight frames by seeding basis matrices.

2.1 Filter Banks

Consider an M -channel filter bank in Fig. 1. Each channel consists of an analysis filter \tilde{h}_m , a synthesis filter h_m ($m = 0, \dots, M - 1$), and down- and upsamplers by N . If $M = N$, the filter bank is called *critically-sampled*; if $M > N$, it is *oversampled*. We assume that all analysis and synthesis filters $\tilde{h}_m = (\tilde{h}_{m,0}, \dots, \tilde{h}_{m,L-1})$ and $h_m = (h_{m,0}, \dots, h_{m,L-1})$ have the same length $L = qN$ for some $q \in \mathbb{N}$ (this assumption is not restrictive as long as all filters have finite support). The operation of the filter bank on a signal x can be described as matrix-vector products (1): the transform $X = \tilde{\Phi}^* x$ is filtering followed by downsampling and the inverse transform $x = \Phi X$ is upsampling followed by filtering. Φ has the form

$$\Phi = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & \Phi_0 & 0 & \dots & 0 & 0 & \dots \\ \dots & \Phi_1 & \Phi_0 & \dots & 0 & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & \Phi_{q-1} & \Phi_{q-2} & \dots & \Phi_0 & 0 & \dots \\ \dots & 0 & \Phi_{q-1} & \dots & \Phi_1 & \Phi_0 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ where } \Phi_r = \begin{pmatrix} h_{0,rN} & \dots & h_{M-1,rN} \\ \vdots & \ddots & \vdots \\ h_{0,rN+N-1} & \dots & h_{M-1,rN+N-1} \end{pmatrix} \in \mathbb{R}^{N \times M} \quad (2)$$

for $0 \leq r \leq q - 1$. Note that the synthesis filters form the columns of Φ .

For (1) to hold, we must have $\Phi \tilde{\Phi}^* = I$. In this paper, we only consider *self-dual* bases and frames, meaning $\Phi = \tilde{\Phi}$ or $\Phi \Phi^* = I$. We can rewrite it in the z -domain using the $N \times M$ polyphase matrix $\Phi_p(z)$ as*

$$\Phi_p(z) = \sum_{r=0}^{q-1} \Phi_r z^{-r}, \quad (3)$$

with Φ_r as defined in (2). We say $\Phi_p(z)$ has degree $q - 1$, since any polynomial in $\Phi_p(z)$ has the degree at most $q - 1$. Using (3), the condition $\Phi \Phi^* = I$ is equivalent to requiring that $\Phi_p(z)$ be paraunitary:

$$\Phi_p(z) \Phi_p^*(z) = I. \quad (4)$$

*The subscript p will always denote a polyphase matrix in this paper and should not be confused with subscripts denoting submatrices as in (2).

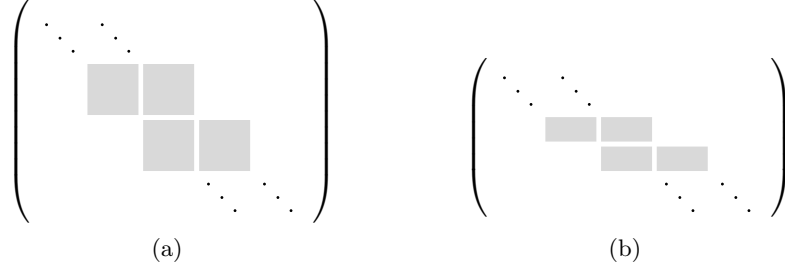


Figure 2. The infinite matrix Φ in (2) for lapped transforms with $q = 2$. The columns of Φ are the basis/frame vectors. (a) $\Phi = \Psi$ for a LOT; (b) Φ for a LTFBT.

Here, $\Phi_p^*(z)$ represents the Hermitian transpose of the polyphase matrix $\Phi(z)$: coefficients are complex-conjugated, z^{-1} is replaced by z , and the matrix is transposed.

If we consider the columns of Φ as vectors in $\ell^2(\mathbb{Z})$, then (4) requires these vectors to form either an orthonormal basis (for $M = N$) or a tight, self-dual frame (for $M > N$) in $\ell^2(\mathbb{Z})$. Then $X = \Phi^*x$ is the associated transform that computes the projection coefficients with respect to Φ (1). If $q \geq 2$, Φ processes overlapping blocks of the signal x , thus leading to a *lapped* transform Φ^* . This allows to avoid the *blocking effects* that occur when a signal is processed in non-overlapping blocks, and each block is treated as an independent signal. The corresponding matrices Φ for lapped transforms are visualized in Fig. 2 and discussed next.

By a slight abuse of notation, we use three equivalent representations of filter bank frames interchangeably as convenient and refer to all of them as frames: a) a set of vectors $\{\varphi_i\}_{i \in \mathbb{Z}}$ spanning $\ell^2(\mathbb{Z})$; b) an infinite matrix Φ as in (2); c) a polyphase matrix $\Phi_p(z)$ as in (3). Also, we emphasize the special case of a basis expansion by using Ψ instead of Φ , and ψ instead of ϕ .

2.2 Basis Expansion

In this paper, we focus on LOTs Ψ^* with basis vector support $L = 2N$ ($q = 2$) whose matrix Ψ is visualized in Fig. 2(a). In this case, the only nonzero blocks in (2) are Ψ_0 and Ψ_1 ; hence, (3) yields a polyphase matrix of degree $q - 1 = 1$:

$$\Psi_p(z) = \Psi_0 + z^{-1}\Psi_1. \quad (5)$$

Since $\Psi_p(z)$ is square, (4) is equivalent to

$$\Psi_0\Psi_0^* + \Psi_1\Psi_1^* = I \quad \text{and} \quad \Psi_0\Psi_1^* = \Psi_1\Psi_0^* = 0. \quad (6)$$

We use these conditions later to show that the new transforms we construct are indeed LOTs.

2.3 Frame Expansion

Frame expansions can be computed with oversampled filter banks, similarly to how critically-sampled filter banks compute basis expansions.

The property $\Phi\Phi^* = I$ for frames is called *tightness*.^{15†} Tight frames can be constructed from orthonormal bases using the Naimark theorem:^{16,17}

THEOREM 2.1. *A set $\{\varphi_i\}_{i \in \mathcal{I}}$ is a tight frame in a Hilbert space \mathbb{H} if and only if there exists a Hilbert space $\mathbb{K} \supset \mathbb{H}$ with an orthonormal basis $\{\psi_i\}_{i \in \mathcal{I}}$, such that the orthogonal projection P of \mathbb{K} onto \mathbb{H} satisfies: $P\psi_i = \varphi_i$, for all $i \in \mathcal{I}$.*

One example of an orthogonal projection is the canonical projection which simply omits coordinates. It is called *seeding*.¹⁸ Originally, seeding was applied to finite-dimensional frames. To seed in the infinite-dimensional case, we extend this approach to polyphase matrices $\Psi_p(z)$:

[†]A frame is tight if $\Phi\Phi^* = cI$. However, since c can be pulled into Φ , we consider only $c = 1$.

DEFINITION 2.2. A frame $\Phi_p(z)$ is obtained by seeding from a basis $\Psi_p(z)$, if it is constructed from $\Psi_p(z)$ by preserving only a subset of the rows of $\Psi_p(z)$. This is written as $\Phi_p(z) = \Psi_p(z)[\mathcal{I}]$, where \mathcal{I} is the list of indices of the retained rows.

The following result is a special case of Theorem 2.1:

LEMMA 2.3. Seeding an orthonormal basis (paraunitary) $\Psi_p(z)$ yields a tight frame $\Phi_p(z)$.

Similarly to Section 2.2, we consider frames in $\ell^2(\mathbb{Z})$ with vector support $L = 2N$, as shown in Fig. 2(b). As in (5), the resulting polyphase matrix $\Phi_p(z)$ has degree 1:

$$\Phi_p(z) = \Phi_0 + z^{-1}\Phi_1,$$

and the tightness condition $\Phi\Phi^* = I$ is equivalent to $\Phi_p(z)$ being paraunitary (4).

LTFTs can be constructed by seeding the polyphase matrix $\Psi_p(z)$ of an LOT basis:⁹ $\Phi_p(z) = \Psi_p(z)[\mathcal{I}]$. In particular, frames in Fig. 2(b) can be constructed by seeding bases in Fig. 2(a). In the following sections, we use seeding to derive new lapped tight frames from new LOT bases. In addition, these frames possess the following properties:⁴

- *Equal norm:* An equal-norm frame has vectors of the same norm, $\|\varphi_i\| = \|\varphi_j\|$, for $i, j \in \mathcal{I}$. Since in the real world, the squared norm of a vector is usually associated with its energy, equal norm is required in situations where equal-energy signals are desirable.
- *Maximal robustness:* An $N \times M$ frame $\Phi_p(z)$ is *maximally robust to erasures*, if and only if any $N \times N$ submatrix of $\Phi_p(z)$ is of full rank on the unit circle. This implies that the loss of up to $M - N$ transform coefficient does not prevent the reconstruction of the signal.

3. CONSTRUCTION OF NEW LOTS AND LTFTS

Our goal is to design real frames that are *tight*, *equal-norm*, and *maximally robust to erasures*. We do this by first constructing LOTS Ψ from a submatrices of the DFT, and then seeding Ψ to obtain the desired frames Φ and corresponding LTFTs Φ^* .

3.1 New Real LOTS

In Section 2.2, we showed that a real LOT basis corresponds to a real square paraunitary polyphase matrix $\Psi_p(z)$ of degree $q - 1$. Although in general $\Psi_p(z)$ is paraunitary if and only if it is unitary on the entire unit circle $|z| = 1$, for a real $\Psi_p(z)$ of degree $q - 1 = 1$ it suffices to check only two conditions:

LEMMA 3.1. Let $\Psi_p(z) = \Psi_0 + z^{-1}\Psi_1$ be a real $M \times M$ polyphase matrix of degree 1, i.e. $\Psi_0, \Psi_1 \in \mathbb{R}^{M \times M}$. $\Psi_p(z)$ is paraunitary if and only if $\Psi_p(1)$ and $\Psi_p(j)$ are unitary.

Proof. “ \Rightarrow ” is immediate. To prove “ \Leftarrow ”, let $\Psi_p(1) = \Psi_0 + \Psi_1$ and $\Psi_p(j) = \Psi_0 - j\Psi_1$ be unitary, i.e.,

$$\begin{cases} (\Psi_0 + \Psi_1)(\Psi_0^T + \Psi_1^T) = I_M \\ (\Psi_0 - j\Psi_1)(\Psi_0^T + j\Psi_1^T) = I_M \end{cases} \Leftrightarrow \begin{cases} \Psi_0\Psi_0^T + \Psi_1\Psi_1^T + \Psi_0\Psi_1^T + \Psi_1\Psi_0^T = I_M \\ \Psi_0\Psi_0^T + \Psi_1\Psi_1^T + j(\Psi_0\Psi_1^T - \Psi_1\Psi_0^T) = I_M \end{cases}. \quad (7)$$

Subtracting the two equations yields

$$\begin{aligned} & \Psi_0\Psi_1^T + \Psi_1\Psi_0^T - j(\Psi_0\Psi_1^T - \Psi_1\Psi_0^T) = 0_M \\ \Leftrightarrow & \Psi_0\Psi_1^T + \Psi_1\Psi_0^T = 0_M \text{ and } \Psi_0\Psi_1^T - \Psi_1\Psi_0^T = 0_M \\ \Leftrightarrow & \Psi_0\Psi_1^T = 0_M \text{ and } \Psi_1\Psi_0^T = 0_M \end{aligned}$$

Inserting into (7) yields $\Psi_0\Psi_0^T + \Psi_1\Psi_1^T = I_M$, and hence the requirements (6) are satisfied. \square

As an example, consider the polyphase matrix $\text{DFT}_{p,K}(z)$:

$$\text{DFT}_{p,K}(z) = \frac{1}{\sqrt{K}} \left[\cos \frac{2k\ell\pi}{K} + z^{-1} \sin \frac{2k\ell\pi}{K} \right]_{0 \leq k, \ell \leq K-1}. \quad (8)$$

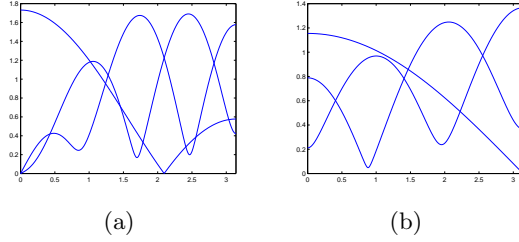


Figure 3. Magnitude Fourier transforms of (a) LOT basis vectors in (9) and (b) seeded LTFT frame vectors in (10) derived using our construction method. The corresponding filter bank has $M = 3$ filters with support $2M = 6$ (LOT) or $2N = 4$ (LTFT).

Both $\text{DFT}_{p,K}(j) = \text{DFT}_K$ and $\text{DFT}_{p,K}(1) = \text{DHT}_K$ (the discrete Hartley transform¹⁹) are unitary; hence, by Lemma 3.1, $\text{DFT}_{p,K}(z)$ is paraunitary.

Next, we show that specific submatrices of $\text{DFT}_{p,K}(z)$ are the LOT matrices $\Psi_p(z)$. The proof of the following theorem is in Appendix A.

THEOREM 3.2. *Let $\Psi_p(z)$ be an $M \times M$ submatrix of $\sqrt{K/M} \text{DFT}_{p,K}(z)$, $K \geq M \geq 2$, constructed by selecting the rows with indices $(r+kR \bmod K \mid 0 \leq k \leq M-1)$ and columns with indices $(c+\ell C \bmod K \mid 0 \leq \ell \leq M-1)$. Here, $0 \leq r, c, R, C < K$.*

Then, $\Psi_p(z)$ is paraunitary if $K = M \gcd(K, RC)$ and K divides $2rC$, $4rc$, and $2MRc$.

Each paraunitary matrix $\Psi_p(z)$ obtained with Theorem 3.2 defines a basis Ψ ; the associated LOT is Ψ^* . In addition, since $\text{DFT}_{p,K}(z)$ is symmetric, we can interchange the lists of row and column indices:

COROLLARY 1. *Let $\Psi_p(z)$ be constructed as in Theorem 3.2. Then, it is paraunitary if $K = M \gcd(K, RC)$ and K divides $2cR$, $4rc$, and $2MCr$.*

As an example, consider a 3×3 submatrix of $\text{DFT}_{p,6}(z)$ constructed as in Theorem 3.2 with parameters $K = 6, M = 3, R = 1, C = 4, r = 0, c = 0$:

$$\Psi_p(z) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}z^{-1} & -\frac{1}{2} + \frac{\sqrt{3}}{2}z^{-1} \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}z^{-1} & -\frac{1}{2} - \frac{\sqrt{3}}{2}z^{-1} \end{pmatrix}. \quad (9)$$

Since the parameters satisfy the theorem, $\Psi_p(z)$ is paraunitary and hence corresponds to an LOT. Fig. 3(a) depicts the magnitude Fourier transforms of the basis vectors in this example.

Next, let us discuss the construction of LTFTs by seeding the above LOTs.

3.2 New Real LTFTs

In this section we seed the $M \times M$ LOTs $\Psi_p(z)$, constructed by Theorem 3.2, to obtain $N \times M$ frames $\Phi_p(z)$ and discuss their properties.

Tightness. By Lemma 2.3, any seeding of $\Psi_p(z)$ obtained with Theorem 3.2 yields a tight frame $\Phi_p(z)$.

Equal norm. Every element of $\Psi_p(z)$ constructed with Theorem 3.2 has the norm $1/\sqrt{M}$. Hence, the columns of any seeded $N \times M$ matrix $\Phi_p(z)$ also have the same norm $\sqrt{N/M}$.

Maximally robust frames. In general, maximal robustness for frames is a property difficult to prove since one has to check that every $N \times N$ submatrix of $\Phi_p(z)$ is invertible. However, it is sufficient to ensure that each such submatrix is nonsingular for at least one value:²⁰

LEMMA 3.3. *An $M \times M$ matrix $\Psi_p(z)$ is nonsingular if and only if there exists $z_0 \in \mathbb{C}$ such that $\det \Psi_p(z_0) \neq 0$.*

THEOREM 3.4. *Let $\Psi_p(z)$ be a paraunitary polyphase matrix constructed as in Theorem 3.2, such that $R \in \{1, K-1\}$. Let the $N \times M$ matrix $\Phi_p(z) = \Psi_p(z)[\mathcal{J}]$, be consecutively seeded from $\Psi_p(z)$, i.e. $\mathcal{J} = (j_0, j_0 + 1, \dots, j_0 + N - 1)$ Then $\Phi_p(z)$ is maximally robust to erasures*

Proof. Consider any $N \times N$ submatrix $\widehat{\Phi}_p(z)$ of matrix $\Phi_p(z)$ constructed as above. Then $\widehat{\Phi}_p(j)$ is a submatrix of DFT_K constructed from its N consecutive columns, and hence it is nonsingular.¹⁸ By Lemma 3.3, $\widehat{\Phi}_p(z)$ is invertible. Thus, any $N \times N$ submatrix of $\Phi_p(z)$ is invertible, and $\Phi_p(z)$ is maximally robust. \square

As an example, consider seeding the matrix $\Psi_p(z)$ in (9). If we retain the first two rows of $\Psi_p(z)$, we obtain

$$\begin{aligned}\Phi_p(z) &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}z^{-1} & -\frac{1}{2} + \frac{\sqrt{3}}{2}z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + z^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \Phi_0 + z^{-1}\Phi_1.\end{aligned}\tag{10}$$

By construction, this frame is tight and equal norm. By Theorem 3.4, the consecutive seeding yields a maximally robust frame; hence, the LTFT in (10) is maximally robust. Fig. 3(b) depicts the magnitude Fourier transforms of the frame vectors in this example.

4. CONCLUSIONS AND DISCUSSION

We presented a construction method to generate new LOTs and LTFTs. The constructed LTFTs are tight and equal-norm; many are maximally robust. We also demonstrated examples of new LOTs and LTFTs. We intend to extend our construction method to the lapped transforms for which the length L of the filters is any integer multiple $q \geq 2$ of M , and in which case, $\Psi_p(z)$ contains polynomials of degree $q - 1$. Another direction of our future work is the generalization of our construction method that will produce a larger class of LOTs and LTFTs. In addition, we will investigate the most general sufficient and necessary conditions on the design of paraunitary submatrices of the DFT.

APPENDIX A. PROOF OF THEOREM 3.2

According to Lemma 3.1, to show that $\Psi_p(z)$ is paraunitary, it is enough to show that $\Psi_p(j)$ and $\Psi_p(1)$ are unitary.

The elements of the matrix $\Psi_p(z)$ are

$$\psi_{k,\ell}(z) = \frac{1}{\sqrt{M}} \left(\cos\left(\frac{2\pi(r+kR)(c+\ell C)}{K}\right) + \sin\left(\frac{2\pi(r+kR)(c+\ell C)}{K}\right)z^{-1} \right),$$

where $0 \leq k, \ell \leq M - 1$ and $0 \leq r, c, R, C \leq M - 1$.

We first find the conditions for $\Psi_p(j)$ to be unitary. The (k, ℓ) -th element of $\Psi_p(j)\Psi_p^*(j)$ is given by

$$\begin{aligned}(\Psi_p(j)\Psi_p^*(j))_{k,\ell} &= \frac{1}{M} \sum_{m=0}^{M-1} \omega_K^{(r+kR)(c+mC) - (r+\ell R)(c+mC)} \\ &= \frac{1}{M} \omega_K^{(k-\ell)RC} \sum_{m=0}^{M-1} \omega_K^{(k-\ell)RCm} \\ &= \begin{cases} 1, & k = \ell; \\ \frac{1}{M} \omega_K^{(k-\ell)RC} \frac{1 - \omega_K^{(k-\ell)RCM}}{1 - \omega_K^{(k-\ell)RC}}, & k \neq \ell. \end{cases}\end{aligned}$$

$\Psi_p(j)$ is unitary if and only if $(\Psi_p(j)\Psi_p^*(j))_{k,\ell} = 0$ for any $k \neq \ell$, or, equivalently, if and only if K is divisible by the product RCM , but not divisible by $(k - \ell)RC$ for any $k - \ell \neq 0$ such that $1 \leq |k - \ell| \leq M - 1$. This is possible if and only if $K = M \text{gcd}(K, RC)$. Thus, $\Psi_p(j)\Psi_p^*(j) = I_M$, and $\Psi_p(j)$ is unitary if and only if $K = M \text{gcd}(K, RC)$.

We next investigate conditions for $\Psi_p(1)$ to be unitary. The m, ℓ th element of $\Psi_p(1)$ is

$$\begin{aligned}\psi_{k,\ell}(1) &= \frac{1}{\sqrt{M}} \left(\cos\left(\frac{2\pi(r+kR)(c+\ell C)}{K}\right) + \sin\left(\frac{2\pi(r+kR)(c+\ell C)}{K}\right) \right) \\ &= \frac{1}{\sqrt{M}} \left(\frac{1+j}{2} \omega_K^{(r+kR)(c+\ell C)} + \frac{1-j}{2} \omega_K^{-(r+kR)(c+\ell C)} \right).\end{aligned}$$

The (k, ℓ) -th element of $\Psi_p(1)\Psi_p^*(1)$ is

$$\begin{aligned}(\Psi_p(1)\Psi_p^*(1))_{k,\ell} &= \frac{1}{M} \sum_{m=0}^{M-1} \left[\left(\frac{1+j}{2} \omega_K^{(r+kR)(c+mC)} + \frac{1-j}{2} \omega_K^{-(r+kR)(c+mC)} \right) \right. \\ &\quad \times \left. \left(\frac{1-j}{2} \omega_K^{-(r+\ell R)(c+mC)} + \frac{1+j}{2} \omega_K^{(r+\ell R)(c+mC)} \right) \right] \\ &= \frac{1}{2M} \sum_{m=0}^{M-1} \left(\omega_K^{(k-\ell)R(c+mC)} + \omega_K^{(\ell-k)R(c+mC)} \right) \\ &\quad + \frac{j}{2M} \sum_{m=0}^{M-1} \left(\omega_K^{(2r+(k+\ell)R)(c+mC)} - \omega_K^{-(2r+(k+\ell)R)(c+mC)} \right) \\ &= \frac{1}{2M} \Sigma_{k,\ell}^{(1)} + \frac{j}{2M} \Sigma_{k,\ell}^{(2)}.\end{aligned}$$

Since $K = M \gcd(K, RC)$, then for any $0 \leq k, \ell \leq M-1$ with $k \neq \ell$, K is not divisible by $(k-\ell)RC$. Thus

$$\Sigma_{k,\ell}^{(1)} = \begin{cases} \sum_{m=0}^{M-1} 2, & k = \ell; \\ \omega_K^{(k-\ell)RC} \frac{1 - \omega_K^{(k-\ell)RCM}}{1 - \omega_K^{(k-\ell)RC}} & k \neq \ell; \\ + \omega_K^{(\ell-k)RC} \frac{1 - \omega_K^{(\ell-k)RCM}}{1 - \omega_K^{(\ell-k)RC}}, & k \neq \ell; \end{cases} = \begin{cases} 2M, & k = \ell; \\ 0, & k \neq \ell. \end{cases}$$

To make $\Psi_p(1)$ a unitary matrix, we choose to impose the condition $\Sigma_{k,\ell}^{(2)} = 0$ for any $0 \leq k, \ell \leq M-1$. If K divides $2rC$, $4rc$, and $2MRc$, then for any k, ℓ

$$\begin{aligned}\Sigma_{k,\ell}^{(2)} &= \omega_K^{(2r+(k+\ell)R)c} \sum_{m=0}^{M-1} \omega_K^{(2r+(k+\ell)R)Cm} - \omega_K^{-(2r+(k+\ell)R)c} \sum_{m=0}^{M-1} \omega_K^{-(2r+(k+\ell)R)Cm} \\ &= \begin{cases} M(\omega_K^{2rc} - \omega_K^{-2rc}), & k + \ell = 0; \\ M(\omega_K^{2rc+MRc} - \omega_K^{-2rc-MRc}), & k + \ell = M; \\ \omega_K^{2rc+(k+\ell)Rc} \frac{1 - \omega_K^{(k+\ell)RCM}}{1 - \omega_K^{(k+\ell)RC}} & \\ -\omega_K^{-2rc-(k+\ell)Rc} \frac{1 - \omega_K^{-(k+\ell)RCM}}{1 - \omega_K^{-(k+\ell)RC}}, & \text{otherwise;} \end{cases} = 0.\end{aligned}$$

Since the above conditions make $\Psi_p(j)$ and $\Psi_p(1)$ unitary, Lemma 3.1 implies that $\Psi_p(z)$ is paraunitary.

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