COMPRESSION OF QRS COMPLEXES USING HERMITE EXPANSION

Aliaksei Sandryhaila[1], *Jelena Kovačević*[2,1], *Markus Püschel*[3]

[1] Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA 15213
[2] Biomedical Engineering and Center for Bioimage Informatics,
Carnegie Mellon University, Pittsburgh, PA 15213
[3] Computer Science, ETH Zürich, 8092 Zürich, Switzerland

ABSTRACT

We propose an algorithm for the compression of ECG signals, in particular QRS complexes, based on the expansion of signals with compact support into a basis of discrete Hermite functions. These functions are obtained by sampling continuous Hermite functions, previously used for the compression of ECG signals. Our algorithm uses the theory of signal models based on orthogonal polynomials, and achieves higher compression ratios compared with algorithms previously reported, both those using Hermite functions, as well as those using the discrete Fourier and discrete cosine transforms.

***Index Terms*—** QRS complex, ECG signal, compression, Hermite function, Hermite transform, DFT, DCT.

1. INTRODUCTION

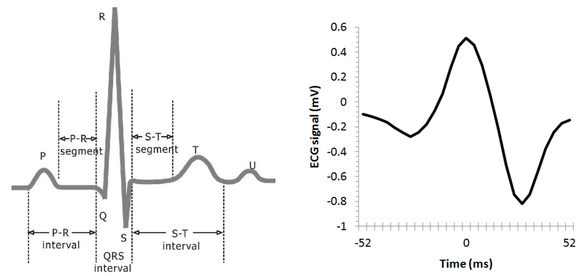
Many signals encountered in electrophysiology often have (or can be assumed to have) a compact support. These signals usually represent the impulse response of a system or organ to an electrical stimulation recorded on the body surface. Examples include electrocardiographic (ECG), electroencephalographic, and myoelectric signals.

Visual analysis of long-term repetitive electrophysiological signals, especially in real time, is a tedious task that requires the presence of a human operator. Computer-based systems have been developed to facilitate this process. For efficient storage, automatic analysis and interpretation, electrophysiological signals are usually represented by a set of features, either heuristic, such as duration and amplitude, or formal, such as coefficients of the expansion in an orthogonal basis. In the latter case, a continuous basis can be used, and the projection and reconstruction of a compact-support signal are computed using numerical methods for integral approximation, such as a numerical quadrature. Alternatively, a discrete basis can be used, and a discrete signal transform, such as the discrete Fourier transform (DFT) or the discrete cosine transform (DCT), can be applied to a digitized signal—obtained by sampling a continuous one.

In both continuous and discrete cases, usually only a few projection coefficients are used for the storage and reconstruction of a signal, leading to a reconstruction error. The goal of the compression optimization is to minimize the error while achieving the greatest compression (for example, by using the fewest coefficients possible).

In this paper, we study the compression of QRS complexes, which are the most characteristic waves of ECG signals [1]. The structure of an ECG signal and an example QRS complex are shown in Fig. 1. In particular, we examine the expansion of QRS complexes into the basis of Hermite functions. Such functions, in their continuous form, provide a highly suitable basis for the representation and compression of QRS complexes [1, 2, 3, 4]. However, as we discuss in Section 3, the reported computer implementations of such expansion suffer from certain limitations, such as the inability to obtain an exact reconstruction of a signal, large computational cost, and an a priori selection of coefficients for reconstruction.

This work was supported in part by NSF grant CCF-633775.



(a) (b)

Fig. 1. (a) ECG structure (image courtesy of National Instruments, http://zone.ni.com/devzone/cda/tut/p/id/6349); (b) QRS complex (centered around the peak).

Contributions. We propose an improved compression algorithm for QRS complexes that expands digitized signals into the basis of *discrete* Hermite functions, obtained by sampling the continuous Hermite functions at specific points, not necessarily on a uniform grid. This approach is based on results obtained from our recently developed theory of signal models based on orthogonal polynomials [5, 6, 7]. The proposed algorithm achieves the perfect reconstruction of signals, has a lower computational cost, and allows us to choose coefficients for reconstruction from a larger pool of coefficients. Experiments comparing the approximation accuracy demonstrate that the new algorithm performs on par with other algorithms for low compression ratios (less than 4.5), and outperforms them for higher compression ratios.

Organization. Section 2 provides an overview of the continuous Hermite functions and their use for the QRS complex compression. It also introduces Hermite polynomial transforms and their properties. Section 3 describes the new compression algorithm and analyzes its advantages. The compression accuracy of the proposed algorithm and other compression algorithms is compared in the experiments discussed in Section 4. Section 5 summarizes the results presented in this paper.

2. BACKGROUND

Hermite functions. Consider the family of polynomials $H_\ell(t)$, $\ell \geq 0$, that satisfy the recursion for $\ell \geq 2$

$$H_\ell(t) = 2tH_{\ell-1}(t) - 2(\ell-1)H_{\ell-2}(t),$$

with $H_0(t) = 1$ and $H_1(t) = 2t$. They are known as *Hermite polynomials*. These polynomials are orthogonal on the real line $\mathbb{R}$

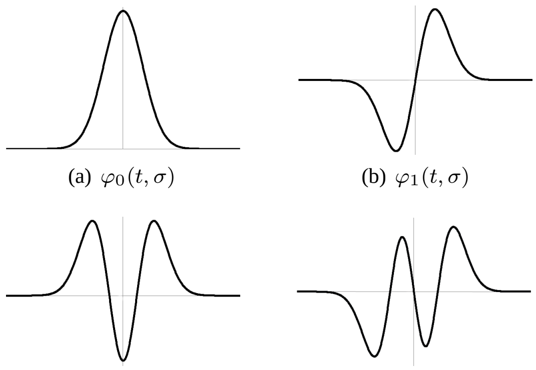


Fig. 2. First four Hermite functions (plotted for the same scale σ).

with respect to the weight function e^{-t^2} :

$$\int_{\mathbb{R}} H_\ell(t)H_m(t)e^{-t^2}dt = 2^\ell \ell! \sqrt{\pi} \cdot \delta_{\ell-m}. \quad (1)$$

It immediately follows from (1) that the functions

$$\varphi_\ell(t,\sigma) = \frac{1}{\sqrt{\sigma 2^\ell \ell! \sqrt{\pi}}} e^{-t^2/2\sigma^2} H_\ell(t/\sigma) \quad (2)$$

are orthonormal on $\mathbb{R}$ with respect to the inner product

$$\langle \varphi_\ell(t,\sigma), \varphi_m(t,\sigma) \rangle = \int_{\mathbb{R}} \varphi_\ell(t,\sigma)\varphi_m(t,\sigma)dt = \delta_{\ell-m}. \quad (3)$$

These functions are called *Hermite functions*. The set of Hermite functions $\{\varphi_\ell(t,\sigma)\}_{\ell\geq 0}$ is an orthonormal basis in the Hilbert space of continuous functions defined on $\mathbb{R}$ [8]. Any such function $s(t)$ can be represented as

$$s(t) = \sum_{\ell \geq 0} c_\ell \varphi_\ell(t,\sigma), \quad (4)$$

where

$$c_\ell = \langle s(t), \varphi_\ell(t,\sigma) \rangle = \int_{\mathbb{R}} s(t)\varphi_\ell(t,\sigma)dt.$$

The first four Hermite functions are shown in Fig. 2. Notice that they become approximately zero as the value of $|t|$ increases. We can assume that they have a compact support, and $\varphi_\ell(t,\sigma) = 0$ for $t \notin [-T_\sigma, T_\sigma]$, where T_σ depends on σ. If $s(t)$ also has a compact support of $[-T_\sigma, T_\sigma]$, then we can compute the coefficients c_ℓ with a finite integral:

$$c_\ell = \int_{\mathbb{R}} s(t)\varphi_\ell(t,\sigma)dt = \int_{-T_\sigma}^{T_\sigma} s(t)\varphi_\ell(t,\sigma)dt. \quad (5)$$

Compression. In practical applications, only a finite number M of Hermite functions are used to reconstruct the signal $s(t)$ in (4). Depending on the computational cost of c_ℓ, the M corresponding coefficients $c_{\ell_0}, \ldots, c_{\ell_{M-1}}$ can be chosen a priori, so that only they are computed. Alternatively, a larger pool of coefficients can be computed, from which M are selected. For an orthonormal basis, it can be demonstrated that selecting coefficients with the largest magnitude minimizes the approximation error, computed as the energy of the difference between the signal $s(t)$ and its approximation with M basis functions.

Digital implementation. A computer-based computation of the coefficient (5) and the Hermite expansion (4) has to be performed in the discrete form. The integral in (5) can be computed with a numerical quadrature using, for example, a rectangle rule:

$$c_\ell = \int_{-T_\sigma}^{T_\sigma} s(t)\varphi_\ell(t,\sigma)dt \approx \sum_{k=-K}^{K} s(\tau_k)\varphi_\ell(\tau_k,\sigma)(t_k - t_{k-1}). \quad (6)$$

Here, $-T = t_{-K-1} < t_{-K} < \ldots < t_{K-1} < t_K = T$, and each $t_{k-1} \leq \tau_k \leq t_k$. The signal is then approximated with M Hermite functions as

$$\hat{s}(\tau_k) = \sum_{m=0}^{M-1} c_{\ell_m}\varphi_{\ell_m}(\tau_k,\sigma). \quad (7)$$

Let t_k be such that $t_k - t_{k-1} = \Delta$ for all k. Then (6) and (7) can be expressed in the matrix-vector notation. Let

$$\mathbf{s} = \begin{pmatrix} s(\tau_{-K}) \\ \vdots \\ s(\tau_K) \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_0 \\ \vdots \\ c_{M-1} \end{pmatrix}, \quad \hat{\mathbf{s}} = \begin{pmatrix} \hat{s}(\tau_{-K}) \\ \vdots \\ \hat{s}(\tau_K) \end{pmatrix}.$$

Then

$$\mathbf{c} = \Delta\Phi^T\mathbf{s} \quad \text{and} \quad \hat{\mathbf{s}} = \Phi\mathbf{c}, \quad (8)$$

where $\Phi \in \mathbb{R}^{(2K+1)\times M}$, such that its m-th column is the ℓ_m-th Hermite function sampled at the points $\tau_{-K}, \tau_{-K+1}, \ldots, \tau_K$:

$$\Phi_{k,m} = \varphi_{\ell_m}(\tau_k,\sigma)$$

for $-K \leq k \leq K, 0 \leq m < M$.

Observe that for perfect reconstruction $\hat{\mathbf{s}} = \mathbf{s}$, Φ must be an orthogonal matrix: $\Phi\Phi^T = I_{2K+1}$.

Compression of QRS complexes: Previous work. The compression of QRS complexes using the expansion into continuous Hermite functions has been studied in [1, 2, 3, 4]. It was originally motivated by the visual similarity of QRS complexes, centered around their peaks, and Hermite functions, as can be observed from Figs. 1 and 2. Varying the value of σ allows us to "stretch" or "compress" the Hermite functions $\varphi_\ell(t,\sigma)$ to optimally match a given QRS complex.

Since ECG signals are usually available as discrete signals equidistantly sampled at $\tau_k = k\Delta$, previous works used $t_k = \tau_k = k\Delta$ in (8). In addition, they proposed to use only the *first* M Hermite functions $\varphi_0(t,\sigma), \ldots, \varphi_{M-1}(t,\sigma)$ for the approximation of QRS complexes.

Hermite polynomial transforms. In [5, 6, 7], we developed a new class of signal models based on orthogonal polynomials[1]. Due to the lack of space, we omit the discussion of these signal models, and only mention the results that are used in Section 3 to construct an optimized QRS complex compression algorithm.

Consider a set of distinct sample points $\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$ and a set of linearly independent polynomials $P = \{P_0(t), \ldots, P_{n-1}(t)\}$. The $n \times n$ matrix

$$\mathcal{P}_{P,\alpha} = \Big[P_\ell(\alpha_k)\Big]_{0\leq k,\ell<n}, \quad (9)$$

[1]A family of polynomials $\{P_\ell(t)\}_{\ell\geq 0}$ is called *orthogonal*, if they satisfy a recursion of the form $tP_\ell(t) = a_\ell P_{\ell-1}(t) + b_\ell P_\ell(t) + c_\ell P_{\ell+1}(t)$, usually with initial conditions $P_0(t) = 1$ and $P_{-1} = 0$. Each family is orthogonal over an interval $I \subseteq \mathbb{R}$ with a weight function $w(t)$: $\int_I P_\ell(t)P_m(t)w(t)dt = 0$ if $\ell \neq m$. Each polynomial $P_\ell(t)$ has exactly ℓ simple real roots. Hermite polynomials are an example of orthogonal polynomials.

is called a *polynomial transform*. In general, it is non-trivial to compute the inverse $\mathcal{P}_{P,\alpha}^{-1}$. However, in the particular case when $P_\ell(t) = \frac{1}{\sqrt{2^\ell \ell!}} H_\ell(t)$ for $0 \le \ell \le n$ are scaled Hermite polynomials, and $\alpha_0, \dots, \alpha_{n-1}$ are the roots of $P_n(t)$,

$$\mathcal{P}_{P,\alpha}^{-1} = \mathcal{P}_{P,\alpha}^{T} D, \tag{10}$$

where $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix whose k-th diagonal element is $\sqrt{2/n}/P_{n-1}(\alpha_k)P_n'(\alpha_k)$.

Using the decomposition algorithm for polynomial transforms derived in [9, 6], a matrix-vector product with $\mathcal{P}_{P,\alpha}$ can be computed with only $3n + O(\frac{n}{2}\log_2^2 \frac{n}{2})$ operations instead of $O(n\log_2^2 n)$, which is the best known computational cost [10]. As a result, the cost is reduced approximately by a factor of 2. Similarly, we can use (10) to compute a matrix-vector product with $\mathcal{P}_{P,\alpha}^{-1}$ with only $4n + O(\frac{n}{2}\log_2^2 \frac{n}{2})$ operations instead of $O(n^2)$ operations that are required in general. The resulting reduction of the computational cost is even more significant, especially for large values of n.

3. PROPOSED ALGORITHM

The compression algorithm based on the expansion into continuous Hermite functions has several important limitations. It does not achieve the perfect reconstruction of a signal **s**, since $\Phi\Phi^T \ne I_{2K+1}$ for $\tau_k = k\Delta$. As a result, $\hat{\mathbf{s}}$ will not converge to **s**, regardless of the number M of Hermite functions used for the construction of an approximation. This problem could be solved by setting $M = 2K+1$ and using Φ^{-1} instead of Φ^T to compute **c** in (8). However, the matrix-vector product $\Phi^{-1}\mathbf{s}$ requires $O((2K+1)^2)$ operations. This cost can be prohibitive for large K, and makes this approach impractical. Finally, the solution suggested in previous works, that uses the *first* M Hermite functions, may not be the optimal choice for the construction of $\hat{\mathbf{s}}$ with M basis functions.

Proposed algorithm. In Section 2 the parameter σ was used to "stretch" and "compress" the Hermite functions $\varphi_k(t,\sigma)$ relatively to the signal $s(t)$. Alternatively, we can fix $\sigma = 1$, and introduce a parameter λ to "stretch" and "compress" the signal $s(t\lambda)$. In this case the numerical quadrature (6) becomes

$$c_\ell = \int_{-T_\lambda}^{T_\lambda} s(t\lambda)\varphi_\ell(t,1)dt \approx \sum_{k=-K}^{K} s(\tau_k\lambda)\varphi_\ell(\tau_k,1)(t_k - t_{k-1}). \tag{11}$$

Furthermore, we can use different, non-equispaced sampling points. Let $\tau_k = \alpha_{k-K}, -K \le k \le K$, be the roots of the Hermite polynomial $H_{2K+1}(t)$, and define polynomials $P_\ell(t) = \frac{1}{\sqrt{2^\ell \ell!}} H_\ell(t)$. Then Φ in (8) has the form

$$\Phi = \pi^{-1/4} W \mathcal{P}_{P,\alpha},$$

where

$$W = \text{diag}\big(e^{-\alpha_k^2/2}\big)_{0\le k<2K+1}$$

is a diagonal matrix, and $\mathcal{P}_{P,\alpha}$ is given in (9).

Furthermore, if $M = 2K+1$, then it follows from (10) that the columns of Φ form an orthogonal basis:

$$\Phi\Phi^T = \pi^{-1/2} W^2 D^{-1} = \Lambda.$$

In order to account for the vector norms, we pre-multiply the input signal **s** with the weight matrix Λ^{-1}, and thus compute

$$\mathbf{c} = \Phi^T \Lambda^{-1} \mathbf{s}$$

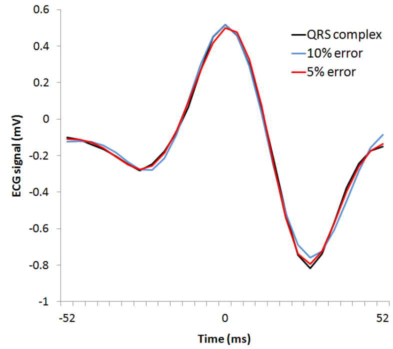


Fig. 3. A QRS complex and its approximations with 10% and 5% errors.

instead of $\mathbf{c} = \Phi^T\mathbf{s}$ in (8).

Advantages. The proposed algorithm addresses all limitations of the original compression algorithms based on continuous Hermite functions. The perfect reconstruction of signals can be achieved by setting $M = 2K+1$ and using $\Phi^T\Lambda^{-1}$ instead of Φ^T in (8) to compute **c**. Since the computational cost of both Φ and Φ^T is approximately $O(\frac{2K+1}{2}\log_2^2(\frac{2K+1}{2}))$, we can compute all coefficients c_ℓ for $0 \le \ell < 2K+1$, and only after that select a few optimal ones to construct the approximation $\hat{\mathbf{s}}$.

4. EXPERIMENTS

Setup. In order to analyze the performance of the proposed compression algorithm, we study the compression of QRS complexes extracted from ECG signals obtained from the MIT-BIH ECG Compression Test Database [11]. A total of $N = 29$ QRS complexes are used. Each complex is available as a discrete signal of length $2K+1 = 27$, and represents a continuous signal of duration 104 milliseconds sampled at 250 Hz.

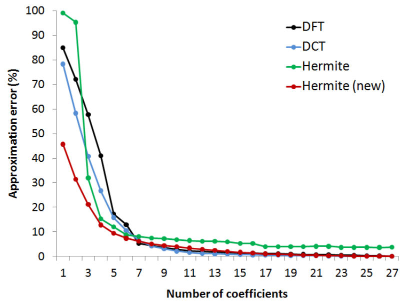
For the original compression algorithm that uses continuous Hermite functions, we compute $2K+1$ coefficients $c_0, \dots, c_{26}$. Among them, we select $1 \le L \le 27$ coefficients with the largest magnitude, construct the approximation $\hat{\mathbf{s}}$ using the transpose of Φ, and compute the approximation error

$$E_L = \frac{||\hat{\mathbf{s}} - \mathbf{s}||_2}{||\mathbf{s}||_2}.$$

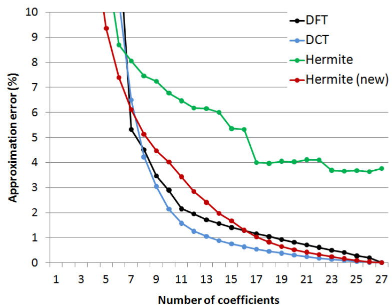
For the new compression algorithm, we have to re-sample the QRS complexes at points $\tau_k\lambda$ proportional to the roots τ_k of $P_{2K+1}(t)$. To do so, we interpolate the available discrete signals with *sinc* functions, and sample it at points $\tau_k\sigma$. Then we compute $2K+1$ coefficients, select L ones with the largest magnitude, construct the approximation using the inverse transform, and compute the approximation error.

In addition, we study the accuracy of compression algorithms based on two widely-used orthogonal discrete signal transforms—DFT and DCT. As above, we apply the transforms to **s**, select L largest coefficients, and compute the approximation error of the reconstruction $\hat{\mathbf{s}}$.

The purpose of the experiment is to obtain average approximation errors of 10% and 5% with the fewest coefficients possible. This goal is motivated by the observation that an approximation that captures 90% of the energy of a QRS complex is sufficient to represent its large-scale features. An approximation that captures 95% of the



(a) All errors.



(b) Errors less than 10%.

Fig. 4. Average approximation errors for different compression algorithms.

energy is sufficiently similar to the original to be used for correct analysis and interpretation, and does not introduce artifacts into the QRS complex. Fig. 3 gives an example of such approximations.

Results. The average approximation errors that were computed during the experiments are plotted in Fig.4. To obtain the average reconstruction error of 10%, our algorithm requires only $L = 5$ coefficients out of $2K + 1 = 27$ (compression ratio 5.4), while the original Hermite algorithm requires 6 coefficients, and DFT and DCT-based algorithms require 7 coefficients (4.5 and 3.86, respectively). To obtain the error of 5%, our algorithm, as well as the ones based on DFT and DCT, requires 8 coefficients (compression ratio 3.5), while the original Hermite algorithm requires 17 coefficients (1.6).

Discussion. As we observe from Fig.4, the new compression algorithm has the lowest approximation error among all algorithms if the compression ratio is 4.5 or higher; i.e. if we use up to 6 out of 27 coefficients for reconstruction. For lower compression ratios, it performs on par with the algorithms based on DFT and DCT, and significantly outperforms the original Hermite algorithm.

The choice of the values for parameters σ and λ is crucial for optimal representation of signals. We have obtained the best results using $\sigma = \lambda = 0.017$ for all $N = 29$ test signals (these values are for variables t and τ_k measured in seconds). However, in computer-based systems these parameters can be adjusted automatically for each ECG signal to achieve a yet higher accuracy of compression and approximation.

5. CONCLUSIONS

We have constructed a new algorithm for the compression of QRS complexes. The proposed algorithm is based on the expansion of signals with compact support (such as ECG signals) into the basis of discrete Hermite functions sampled from continuous Hermite functions at sampling points that are the roots of a corresponding Hermite polynomial. The new method uses results from our recently developed theory of signal models for orthogonal polynomials, and achieves a better compression ratio than the original algorithm based on continuous Hermite functions. In addition, the computational cost of the compression and approximation is reduced.

6. REFERENCES

[1] P. Laguna, R. Jané, S. Olmos, N. V. Thakor, H. Rix, and P. Caminal, "Adaptive estimation of QRS complex wave features ECG signal by the Hermite model," *J. of Med. and Biol. Eng. and Comput.*, vol. 34, no. 1, pp. 58–68, 1996.

[2] L. Sörnmo, P. O. Börjesson, P. Nygards, and O. Pahlm, "A method for evaluation of QRS shape features using a mathematical model for the ECG," *IEEE Trans. on Biomed. Eng.*, vol. BME-28, no. 10, pp. 713–717, 1981.

[3] L. R. Lo Conte, R. Merletti, and G. V. Sandri, "Hermite expansion of compact support waveforms: Applications to myoelectric signals," *IEEE Trans. on Biomed. Eng.*, vol. 41, no. 12, pp. 1147–1159, 1994.

[4] M. Lagerholm, C. Peterson, G. Braccini, L. Edenbranhdt, and L. Sörnmo, "Clustering ECG complexes using Hermite functions and self-organizing maps," *IEEE Trans. on Biomed. Eng.*, vol. 47, no. 7, pp. 838–848, 2000.

[5] M. Püschel and J. M. F. Moura, "Algebraic signal processing theory," available at http://arxiv.org/abs/cs.IT/0612077, parts of this manuscript are published as [12] and [13].

[6] A. Sandryhaila, *Algebraic Signal Processing: Modeling and Subband Analysis*, Ph.D. thesis, Carnegie Mellon University, Pittsburgh, PA, 2010.

[7] A. Sandryhaila, J. Kovacevic, and M. Püschel, "Algebraic signal processing theory: Generic nearest-neighbor models," in preparation.

[8] A. Jerri, *Integral and Discrete Transforms with Applications and Error Analysis*, CRC Press, 1992.

[9] A. Sandryhaila, J. Kovacevic, and M. Püschel, "Algebraic signal processing theory: Cooley-Tukey type algorithms for polynomial transforms based on induction," submitted for publication.

[10] J. R. Driscoll, D. M. Healy Jr., and D. Rockmore, "Fast discrete polynomial transforms with applications to data analysis for distance transitive graphs," *SIAM Journal Comput.*, vol. 26, pp. 1066–1099, 1997.

[11] "MIT-BIH ECG Compression Test Database," http://www.physionet.org/physiobank/database/cdb.

[12] M. Püschel and J. M. F. Moura, "Algebraic signal processing theory: Foundation and 1-D time," *IEEE Trans. on Signal Proc.*, vol. 56, no. 8, pp. 3572–3585, 2008.

[13] M. Püschel and J. M. F. Moura, "Algebraic signal processing theory: 1-D space," *IEEE Trans. on Signal Proc.*, vol. 56, no. 8, pp. 3586–3599, 2008.