Quantized Frame Expansions as Source–Channel Codes for Erasure Channels

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Abstract

Quantized frame expansions are proposed as a method for generalized multiple description coding, where each quantized coefficient is a description. Whereas previous investigations have revealed the robustness of frame expansions to additive noise and quantization, this represents a new application of frame expansions. The performance of a system based on quantized frame expansions is compared to that of a system with a conventional block channel code. The new system performs well when the number of lost descriptions (erasures on an erasure channel) is hard to predict.

1 Introduction

The problem of transmitting data over heterogenous networks has recently received considerable attention. A typical scenario might require data to move from a fiber link to a wireless link, which necessitates dropping packets to accommodate the lower capacity of the latter. If the network is able to provide preferential treatment to some packets, then the use of a multiresolution or layered source coding system is the obvious solution. But what if the network will not look inside packets and discriminate? Then packets will be dropped at random, and it is not clear how the source (or source–channel) coding should be designed. If packet retransmission is not an option (*e.g.*, due to a delay constraint or lack of a feedback channel), one has to devise a way of getting meaningful information to the recipient despite the loss. The situation is similar if packets are lost due to transmission errors or congestion.

This problem is a generalization of the "multiple description" (MD) problem. In the MD problem, a source is described by two descriptions at rates R_1 and R_2. These two descriptions individually lead to reconstructions with distortions D_1 and D_2, respectively; the two descriptions together yield a reconstruction with distortion D_0. The original problem, as posed by Gersho, Witsenhausen, Wolf, Wyner, Ziv and Ozarow in 1979, was to characterize the achievable quintuples $(R_1, R_2, D_0, D_1, D_2)$. The first design algorithm for practical MD coding was given by Vaishampayan [1] and the first transform-based approach was devised by Wang, Orchard, and Reibman [2]. For more background on MD coding, see [1, 3] and the references therein.

Describing a source with M packets (descriptions) such that any subset of the packets yields a useful reconstruction is a generalization of the MD problem. We propose the use of linear transforms from $\mathbb{R}^N$ to $\mathbb{R}^M$ (with $M > N$) followed by scalar quantization as a computationally simple approach to generalized MD coding. Linear transform approaches for $M = N$ are investigated in [2, 4, 5].

2 Problem Statement

Consider communicating a source taking values in $\mathbb{R}^N$ across an erasure channel. Denote the channel alphabet by $\mathcal{X}$ and suppose $|\mathcal{X}| = 2^{(N/M)R}$, where M is the number of channel uses per N-tuple and R is the overall rate per component (including channel coding). The objective of this paper is to compare the following two techniques:

- **Conventional system:** Each component of the source vector is quantized using an $(N/M)R$-bit quantizer, giving N codewords. A linear systematic block code $\mathcal{X}^N \to \mathcal{X}^M$ is applied to the quantizer output, and the M resulting codewords are sent on the channel.

- **Quantized frame (QF) system:** The source vector is expanded using a linear transform $F : \mathbb{R}^N \to \mathbb{R}^M$. Each transform coefficient is quantized using an $(N/M)R$-bit quantizer. The M resulting codewords are sent on the channel.

Since a linear block code is a linear transform, the difference between the systems is the swapping of transform and quantization operations. The second system uses a QF expansion. For details on frames and QF expansions, see [6, 7].

The conventional system works by producing a linear dependence between the transmitted symbols. A valid transmitted M-tuple must lie in a specified N-dimensional subspace of $\mathcal{X}^M$. When N or more symbols are received, they are consistent with exactly one valid element of $\mathcal{X}^M$, so the information symbols are known. This works very well when exactly N of the M transmitted codewords are received. However, when more or less than N codewords are received, there either is no benefit from the extra information or it is difficult to recover partial information about the source vector.

The QF system has a similar way of adding redundancy. Denote the signal vector by x. The expanded signal $y = Fx$ has a linear dependence between its components. Thus, if N or more components of y are known, x can be recovered exactly. However, the components of y are not directly transmitted; it is the quantization that makes the two systems different. Quantization makes the components of $\hat{y} = Q(y)$ linearly independent, so each component of $\hat{y}$—even in excess of N—gives distinct information on the value of x. It is known that from a source coding point of view, a QF expansion (with linear reconstruction) is not competitive with a basis expansion [3, 7, 8]. Here the "baseline" fidelity is given by a basis expansion, and the noise reduction property of frames [7] improves the fidelity when we are "lucky" to receive more than N components.

One should not get the impression that the QF system is automatically as good as the conventional system when N components are received and better when more than N are received. The comparison is more subtle because all basis expansions are not equally good. The conventional system can use the best orthogonal basis (the Karhunen–Loève transform) or at least an orthogonal basis. On the other hand, it is not possible to make all N-element subsets of the frame associated with F orthogonal. Quantizing in a nonorthogonal basis is inherently suboptimal [9].

When less than N components are received, the QF representation fails to localize x to a finite cell. Neglect quantization error for the moment, and assume $k < N$ components are received. $\mathbb{R}^N$ can be decomposed into a k-dimensional subspace and an $(N-k)$-dimensional perpendicular subspace such that the component of x in the k-dimensional subspace is completely specified and the component in the perpendicular subspace is unknown. In many applications the source is known to have mean zero, so the component in the perpendicular subspace can be estimated as zero. Thus the reconstruction process may follow the same linear algebraic calculations for any $k < N$ received components without a distinction between systematic and parity parts of the code.

We examine the QF system in two steps: First, we assume that the quantization error is additive white noise, independent of the source, and that a linear reconstruction is used. Then, we consider the communication of a white Gaussian source and fix the quantization to be unbounded and uniform. This facilitates a specific numerical comparison between the conventional and QF systems using the earlier analysis.

3 Effect of Erasures

Let $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{R}^N$ be a tight frame with $\|\varphi_k\| = 1$ for all k, and let F be the frame operator associated with Φ.[1] Φ being a frame means that there exist A and B, $0 < A \le B < \infty$, such that

$$A\|x\|^2 \le \sum_{k=1}^{M} \|\langle x,\, \varphi_k\rangle\|^2 \le B\|x\|^2 \qquad \text{for all } x \in \mathbb{R}^N.$$

With the normalization and tightness of the frame, $A = B = M/N$. The frame operator F is given by a matrix with kth row equal to φ_k^*. A source vector $x \in \mathbb{R}^N$ is represented by $\hat{y} = Q(y)$, where $y = Fx$ and Q is a scalar quantizer. Since the components of $\hat{y}$ will be used as "descriptions" in a multiple description system, we are interested in the distortion incurred in reconstruction from a subset of the components of $\hat{y}$.

We will model $\eta = \hat{y} - y$ as white noise independent of x with component variances σ_η^2. This is a common model, though η is actually completely determined by x when Q is a deterministic quantizer. If subtractive dithered uniform quantization with step size Δ is used, the model is precisely valid with $\sigma_\eta^2 = \Delta^2/12$ [10]. We will ignore the

[1] Definitions of frame terminology are kept to a minimum; the reader is referred to [6] for details.

distribution of the quantization error and use linear reconstruction strategies that minimize the residual $\|\hat{y} - F\hat{x}\|_2$.

Each of the M channels of the multiple description system carries one coefficient $\langle x, \varphi_k \rangle + \eta_k$. If there are no erasures, the linear reconstruction that minimizes the MSE uses the dual frame $\widetilde{\Phi} = (N/M)\Phi$ [6]; the resulting reconstruction error is given by

$$\mathrm{MSE}_0 \;=\; E[N^{-1}\|x - \hat{x}\|^2] \;=\; \frac{N}{M}\sigma_\eta^2. \tag{1}$$

3.1 Effect of Erasures on the Structure of a Frame Suppose now that some of the descriptions are lost. Let E denote the index set of the erasures, so $\hat{y}_k$ for $k \in E$ are lost. The number of erasures is denoted $e = |E|$. The description at the decoder is an expansion with respect to $\Phi' = \Phi \setminus \{\varphi_k\}_{k \in E}$. If Φ' is a frame then, under the assumptions on the quantization noise, the best linear reconstruction uses the frame dual to Φ' [6].

When is Φ' a frame? The following proposition shows that when one element of a normalized tight frame is deleted, the remaining set of vectors is a frame, but not a tight frame:[2]

Proposition 1 *Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ be a tight frame with $\|\varphi_k\| = 1$ for all k. For any i, $\Phi' = \Phi \setminus \varphi_i$ is a frame. Φ' has lower frame bound $A' = M/N - 1$ and upper frame bound $B' = M/N$.*

Proposition 1 can be extended to more erasures if M/N is large. Specifically, e erasures will leave a frame when $M/N > e$. This is a far cry from being able to guarantee that $M - N$ erasures leaves a basis for $\mathbb{R}^N$. Fortunately, there exist families of frames for which this is true. One such family is the harmonic frames.

Harmonic frames, or Fourier frames, give overcomplete discrete Fourier expansions. A harmonic tight frame is given by

$$\varphi_k[i] \;=\; \frac{1}{\sqrt{N}} W_M^{(k-1)(i-1)}, \qquad i = 0, 1, \ldots, N-1, \quad k = 0, 1, \ldots, M-1, \tag{2}$$

where $W_M = e^{j2\pi/M}$ is the Mth root of unity. (A real harmonic tight frame could be defined similarly.) The following result guarantees that after erasing up to $M - N$ elements from a harmonic tight frame, we still have a frame.

Proposition 2 *Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{C}^N$ be a harmonic tight frame, with φ_k as in (2). Then, any subset of N or more vectors from Φ forms a frame.*

3.2 Effect of Erasures on the MSE We now consider the effect of erasures on the MSE. Assume that $\Phi' = \Phi \setminus \{\varphi_k\}_{k \in E}$ is a frame; hence, $e \leq M - N$. Larger numbers of erasures are considered in Section 4. When Φ' is not a frame, x can only be estimated to within a subspace and distributional knowledge is needed to get a good estimate.

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[2] Proofs of Propositions 1 and 2 are omitted due to lack of space.

When any one element of Φ is erased, the MSE is given by

$$\mathrm{MSE}_1 \;=\; \left(1+\frac{1}{M-N}\right)\frac{N}{M}\sigma_\eta^2 \;=\; \left(1+\frac{1}{M-N}\right)\mathrm{MSE}_0. \tag{3}$$

This result can be obtained by averaging the power of the quantization noise projected on to $\mathbb{R}^N$ by the frame dual to Φ'.[3]

Assume now that there are e erasures. Let φ be the $N \times e$ matrix comprised of the erased components and let $P_\varphi = \varphi^*\varphi$. Let

$$A^{(e)} = \left(I_e - \frac{N}{M}P_\varphi\right)^{-1}, \quad B^{(e)} = 2A^{(e)} + \frac{N}{M}A^{(e)}P_\varphi A^{(e)}, \quad C^{(e)} = \left(\frac{M}{N}P_\varphi - P_\varphi P_\varphi\right)^T.$$

Then the MSE with e erasures is

$$\mathrm{MSE}_e \;=\; \left[1 - \frac{e}{M} + \frac{N}{M^2}\sum_{i,j=1}^{e} B_{ij}^{(e)} C_{ij}^{(e)}\right]\mathrm{MSE}_0. \tag{4}$$

A simple special case is when the erased components are pairwise orthogonal. In this case, $P_\varphi = I_e$ and MSE_e reduces to

$$\mathrm{MSE}_e \;=\; \left(1+\frac{e}{M-N}\right)\mathrm{MSE}_0. \tag{5}$$

The analysis presented thus far makes no assumptions about the source and instead makes strong assumptions about the quantization error. In effect, it is a distortion-only analysis; since the source has not entered the picture, there is no relationship between σ_η^2 and the rate. This is remedied in the following section.

4 Performance Analysis and Comparisons

Let x be a zero-mean, white, Gaussian vector with covariance matrix $R_x = \sigma^2 I_N$. This source is convenient for analytical comparisons between the QF system and a conventional communication system that combines scalar quantization with a block channel code. Entropy-coded uniform quantization (ECUQ) will be used in both systems. The distortion–rate performance of ECUQ on a Gaussian variable with variance σ^2 will be denoted $D_{\sigma^2}(R)$. This function directly gives the performance of the conventional system when the channel code is successful in eliminating the effect of erasures and is also useful in describing the performance of the QF system.

4.1 Performance of the Conventional System We first analyze the conventional system. For coding at a total rate of R bits per component of x (including channel coding), NR bits are split among M descriptions. Thus the overall average distortion per component with e erasures is

$$\bar{D}_e \;=\; D_{\sigma^2}\left(\frac{NR}{M}\right), \qquad \text{for} \quad e \;=\; 0, 1, \ldots, M-N. \tag{6}$$

[3] For complete derivations of (3), (4), and (5), see [3].

When $e > M - N$, the channel code cannot correct all of the erased information symbols. Since the code is systematic, the decoder will have received some number of information symbols and some number of parity symbols. Assume that the decoder discards the parity symbols and estimates the erased information symbols by their means. Denoting the number of erased information symbols by e_s, the average distortion per component for e_s erased information symbols is then

$$\widetilde{D}_{e_s} \;=\; \frac{e_s}{N}\sigma^2 + \frac{N-e_s}{N} D_{\sigma^2}\Big(\frac{NR}{M}\Big), \qquad \text{for} \quad e \;=\; M-N+1,\,\ldots,\,M-1,\,M. \quad (7)$$

As it is, (7) does not completely describe the average distortion because the relationship between e and e_s is not specified. In fact, given that there are e total erasures, e_s is a random variable. There are $\binom{M}{e}$ ways that e erasures can occur and we assume these to be equally likely. The probability of k erased information symbols is then

$$P\Big(e_s = k \;\Big|\; e-(M-N) \le k \le \min(e,N)\Big) \;=\; \binom{M}{e}^{-1}\binom{N}{k}\binom{M-N}{e-k}.$$

Using this gives the average distortion per component as

$$\begin{aligned}\bar{D}_e \;&=\; \sum_{e_s=e-(M-N)}^{\min(e,N)} P(e_s = k \mid e \text{ total erasures})\,\widetilde{D}_{e_s} \\ &=\; \binom{M}{e}^{-1} \sum_{e_s=e-(M-N)}^{\min(e,N)} \binom{N}{e_s}\binom{M-N}{e-e_s}\left[\frac{e_s}{N}\sigma^2 + \frac{N-e_s}{N}D_{\sigma^2}\Big(\frac{NR}{M}\Big)\right] \quad (8)\end{aligned}$$

for $e = M-N+1, \ldots, M-1, M$, because the received components of x are subject to quantization error and the erased components have variance σ^2.

There is no denying that discarding the parity symbols is not the optimal reconstruction strategy—to minimize MSE or probability of error. However, it comes close to minimizing the MSE; actually minimizing the MSE seems computationally difficult. Investigation of a couple of cases provides a credible justification for discarding the parity information. Consider $e = M - N + 1$, one more erasure than can be corrected. One extreme case is $e_s = 1$, where all the parity symbols are erased. In this case there is no parity information, so estimating the erased information symbol by its mean is clearly the best that can be done. In the other extreme case, $e_s = e$ and all the parity information is received. For convenience, number the erased components so that $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_e$ are lost. If a single one of these were known, then the rest could be determined because the code can correct $e - 1$ erasures. So a possible decoding method is as follows: For each possible value of $\hat{x}_1$, determine $\hat{x}_2, \hat{x}_3, \ldots,$ $\hat{x}_e$. Since the $\hat{x}_i$'s are independent, it is easy to compute the probabilities of each of the $[\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_e]^T$ vectors. The centroid of the vectors gives the minimum MSE estimate.

There are two main difficulties with this computation. Firstly, the number of possibilities to enumerate is exponential in $e-(M-N)$—namely $|\mathcal{X}|^{e-(M-N)}$, where $\mathcal{X}$ is the alphabet for $\hat{x}$—and may be very large. More importantly, it may simply not be useful to compute the probability density of the possible vectors. The nature of the channel code is to make values in each possible $[\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_e]^T$ vector more or less uniform. Thus the minimum MSE estimate is often close to simply estimating each component by its mean.

4.2 Performance of the QF System When F is the frame operator associated with a normalized tight frame Φ, each component of $y = Fx$ is still Gaussian with mean zero and variance σ^2. Thus D_{σ^2}, as defined previously, can again be used to describe the distortion–rate characteristics of the quantized coefficients $\hat{y}_i$'s. These distortions, however, do not equal the component distortions in the reconstruction of x because of the use of frames and nonorthogonal bases.

We assume the frame is designed such that all subsets of at least N elements form a frame, as with harmonic frames (see Proposition 2). Then when there are at most $M-N$ erasures, we can approximate the distortion using (4). Specifically, using (1) and noting that D_{σ^2} connects the coding rate to the quantization noise power σ_η^2, we obtain

$$\bar{D}_0 \;=\; \frac{N}{M} D_{\sigma^2}\Big(\frac{NR}{M}\Big), \tag{9}$$

which is better than the performance of the conventional system. For $e = 1, 2, \ldots, M-N$, there is no simple closed form for the distortion, but it can be written as

$$\bar{D}_e \;=\; c_e D_{\sigma^2}\Big(\frac{NR}{M}\Big), \qquad \text{for} \quad e \;=\; 1,\, 2,\, \ldots,\, M-N. \tag{10}$$

The constant c_e is M/N times the average of the bracketed term of (4), where the average is taken over all possible positions of e erasures. With a given frame, additional measurements always reduce the average reconstruction error (that is, more erasures always increase it), so $\{c_e\}_{e=0}^{M-N}$ is an increasing sequence.

When there are more than $N-M$ erasures, the decoder has less than a basis representation of x. The source vector x can be orthogonally decomposed as

$$x = x_S + x_{S^\perp} \qquad \text{where} \quad x_S \in S = \operatorname{span}(\{\varphi_k\}_{k \notin E}).$$

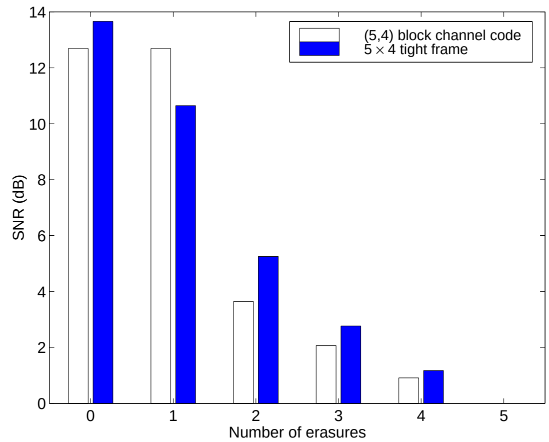
Since the source is white and Gaussian, x_S and $x_{S^\perp}$ are independent. Thus not only does the decoder have no direct measurement of $x_{S^\perp}$, but it has absolutely no way to estimate it aside from using its mean. Estimating $x_{S^\perp} = 0$ introduces a distortion of $N^{-1}(e-(M-N))\sigma^2$ because the dimension of $S^\perp$ is $e-(M-N)$. The received coefficients $\{\hat{y}_k\}_{k \notin E}$ provide a quantized basis representation of x_S. The basis will generally be a nonorthogonal basis, so the per component distortion will exceed $D_{\sigma^2}(NR/M)$ by a constant factor which depends on the skew of the basis. Thus we conclude

$$\bar{D}_e \;=\; \frac{e-(M-N)}{N}\sigma^2 + \frac{M-e}{N} c_e D_{\sigma^2}\Big(\frac{NR}{M}\Big), \quad \text{for} \quad e \;=\; M-N+1,\, \ldots,\, M-1,\, M. \tag{11}$$

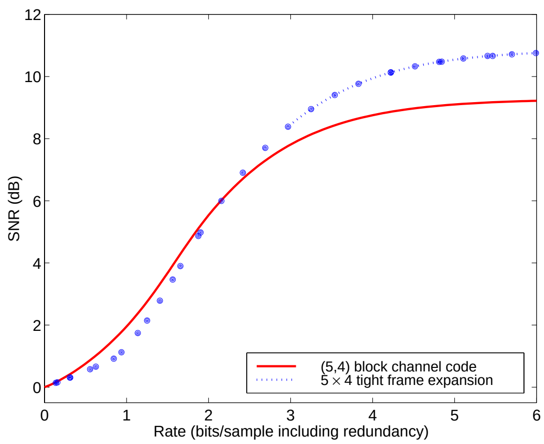
The constant factor c_e is computed in [3] using techniques from [7]. It is always larger than 1, since it is not possible for all subsets of a given size of the frame to be orthogonal.

	Baseline system	QF system
$\bar{D}_0$	$D_{\sigma^2}(\frac{4}{5}R)$	$\frac{4}{5}D_{\sigma^2}(\frac{4}{5}R)$
$\bar{D}_1$	$D_{\sigma^2}(\frac{4}{5}R)$	$\frac{8}{5}D_{\sigma^2}(\frac{4}{5}R)$
$\bar{D}_2$	$\frac{2}{5}\sigma^2 + \frac{3}{5}D_{\sigma^2}(\frac{4}{5}R)$	$\frac{1}{4}\sigma^2 + \frac{9}{10}D_{\sigma^2}(\frac{4}{5}R)$
$\bar{D}_3$	$\frac{3}{5}\sigma^2 + \frac{2}{5}D_{\sigma^2}(\frac{4}{5}R)$	$\frac{2}{4}\sigma^2 + \frac{8}{15}D_{\sigma^2}(\frac{4}{5}R)$
$\bar{D}_4$	$\frac{4}{5}\sigma^2 + \frac{1}{5}D_{\sigma^2}(\frac{4}{5}R)$	$\frac{3}{4}\sigma^2 + \frac{1}{4}D_{\sigma^2}(\frac{4}{5}R)$
$\bar{D}_5$	σ^2	σ^2

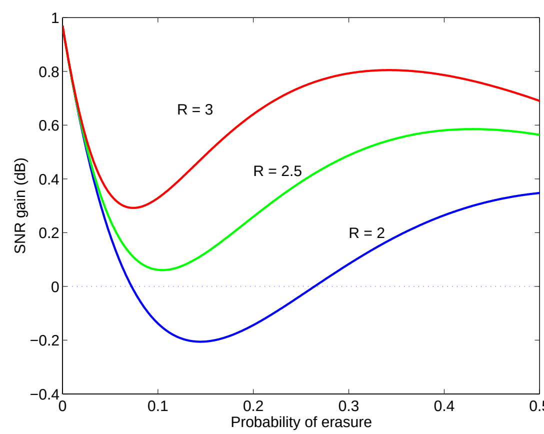
(a)



(b)



(c)



(d)

Figure 1: Comparison between conventional and quantized frame systems for $N = 4$, $M = 5$. (a) The table gives the average distortion–rate performance for each possible number of erasures. (b) Plotted is the SNR for each number of erasures at rate $R = 3$. (c) The channel is assumed to be memoryless with erasure probability 1/5; the SNR is given as a function of total rate. (d) The probability of erasure is varied for three fixed rates; the SNR advantage of the QF system, which may be negative, is shown.

4.3 A Numerical Example Comparing (6) and (8) to (9)–(11) does not immediately reveal the relative merits of the two systems. This section presents a simple numerical example to compare the two systems. This example is developed in greater detail in [3].

Let $N = 4$ and $M = 5$ and let Φ be a 5-element tight frame in $\mathbb{R}^4$. The table in Figure 1 gives distortion expressions based on (6)–(11). These are evaluated at a rate of 3 bits per component, yielding the signal-to-noise ratios shown in the bar graph. At this rate, the QF system is superior except when there is exactly one erasure.

The number of erasures is random. If we assume the erasures are independent, then a single probability of erasure fixes weightings for the distortions of Figure 1(a)–(b). Comparisons at different rates and probabilities of erasure are shown in Figure 1(c)–(d). At moderate-to-high rates, the QF system exhibits a robustness to mismatch between the probability of erasure and the fraction of rate allocated to channel coding.

4.4 Asymptotic Behavior The example presented in Section 4.3 provides some insight into the performance of the QF system, but it is just a single example. At this time, we are unable to make strong statements about the potential of the QF system because the achievable sets of c_e's are unknown. Nevertheless, we may make a few comments on the asymptotic behavior of the QF system. Both high rate and large block length asymptotics are considered.

In the limit of high rate, quantization error is negligible in comparison to the distortion caused by completely missing one orthogonal component of the source. The distortion goes to zero when there are at most $M - N$ erasures, but for larger numbers of erasures the distortion approaches $N^{-1}(e-(M-N))\sigma^2$. Compared to an unconstrained multiple description source coding scheme, the asymptotic performance with more than $M - N$ erasures is very poor. One could use M independent vector quantizers to form the M descriptions. In this case every side distortion would asymptotically approach zero. Such a scheme would presumably have high encoding and decoding complexity (in time, memory, or both); this is why we are interested in linear transform-based approaches.

Comparing the QF system to the conventional system at high rate, the QF system is better when there are more than $M - N$ erasures. In this case, the QF system loses an $e-(M-N)$-dimensional part of the signal while the conventional system loses at least this much; averaging over all erasure patterns, the conventional system loses even more. For lower numbers of erasures, the relative performance depends on the factor c_e. This constant generally depends on the tight frame, but has a simple form in two cases: $c_0 = N/M$ and $c_1 = M^{-1}N(1+(M-N)^{-1})$. It is also known that c_{M-N} must be larger than 1. $\{c_e\}_{e=0}^{M-N}$ is monotonic in the number of erasures and crosses 1 somewhere between $e = 0$ and $e = M - N$.

In information theory it is typical to look at performance limits as the block size grows without bound. In channel coding for a memoryless channel this makes the number of erasures predictable, by the law of large numbers. Using multiple description coding as an abstraction for coding for an erasure channel is in part an attempt to avoid large block sizes and to cope with unpredictable channels. Nevertheless, it is useful understand the performance of the QF system with large block sizes.

A performance analysis must depend in some part on a choice of a set of frames. Intuition suggests that the best frame, at least for a white source, is one that uniformly covers space. The following proposition shows that asymptotically, the most uniform frame approaches an orthonormal basis in a certain sense (for a proof, see [3]):

Proposition 3 *Suppose that $M/N = r$, with $1 < r < 2$. Let $\Phi = \{\varphi_k\}_{k=1}^{M}$ be a frame in $\mathbb{R}^N$. If the design of Φ is the packing of M lines in $\mathbb{R}^N$ such that the minimum angular separation is maximized, then as $N \to \infty$ (M increasing accordingly as $M = \lfloor rN \rfloor$) the elements of Φ become pairwise orthogonal.*

An upper bound on the constant in (10), c_{M-N}, close to 1 would be useful in bounding the worst case performance of the QF system with respect to the conventional system. Proposition 3 suggests that if a frame is designed to maximize uniformity in the specified manner and any $M-N$ elements of the frame are deleted, the remaining set is approximately an orthonormal basis. Unfortunately, the convergence in Proposition 3 does not lead to small bounds on the c_e's. Numerical computations show that as M and N are increased with M/N held constant, the constant factor c_{M-N} increases. This holds for harmonic frames as well as frames designed as in Proposition 3. This negative result on using the QF system with large $M-N$ should not discourage its use in systems with small numbers of descriptions.

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