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# Designing Local Orthogonal Bases on Finite Groups II: Nonabelian Case

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**Abstract**

We extend to general finite groups a well-known relation used for checking the orthogonality of a system of vectors as well as for orthogonalizing a nonorthogonal one. This, in turn is used for designing local orthogonal bases obtained by unitary transformations of a single prototype filter. The first part of this work considered the abelian groups of unitary transformations, while here we deal with nonabelian groups. As an example, we show how to build such bases where the group of unitary transformations consists of modulations and rotations. Such bases are useful for building systems for evaluating image quality.

## 1 Introduction

In the first part of this work [1], we showed how to design local orthogonal bases on finite abelian groups. In this part we make the extension to finite nonabelian groups, and in particular, to the group of modulations and rotations.

To work with such groups, we need to extend the following well-known fact:

$$\langle f(t), f(t-n)\rangle \;=\; \delta(n) \;\Leftrightarrow\; \sum_{n=-\infty}^{\infty} |F(\omega+2\pi k)|^2 = 1, \tag{1}$$

where $f(t)$ is a continuous-time function and $F(\omega)$ is its Fourier transform. Property (1) proves its usefulness both in testing the orthogonality of $f(t)$ with respect to its integer translations as well as in producing functions enjoying such a property. Property (1) can also be seen as a necessary and sufficient condition for the orthogonality of $f$ with respect to the functions obtained by applying to $f$ a group of unitary linear transformations, actually translations by integer values. Note that although the group in 1 is infinite, in this work we study only finite groups. Relation 1 serves only as a guiding light.

As mentioned earlier, in [1] we extended (1) to finite abelian groups. Our aim here is to do the same for the nonabelian groups.

The outline of the paper is as follows: All the preliminaries as well as the abelian case are covered in [1]. Section 2 extends those results to nonabelian groups. Section 3 discusses the filter design problem while Section 4 gives an example of a design where the group consists of modulations and rotations. Appendix A collects the proofs of all of the results in the paper.

# 2 Orthonormal Sets Obtained from Finite Nonabelian Groups of Linear Transformations

Our discussion in [1] on how to build orthonormal sets of vectors obtained by applying finite abelian groups of unitary linear transformations is far from being completely general. Indeed, the group of rotations and modulations that triggered this work is not abelian; thus, the previous theory has to be modified to be able to work with such a group. The main problem that arises with noncommutative groups is that the representations are no longer one dimensional which gives rise to some technical difficulties. Note that in this section we work with the vector sets introduced and formally described in [1], and denoted by bold capital letters.

## 2.1 Fourier Transform as a Tool for Nonabelian Groups

Consider the definition of the Fourier transform for nonabelian groups as given in Appendix A of [1]. We want to apply the Fourier transform given by (65) in Part I [1] to the condition of $\{U\boldsymbol{B} \mid U \in \Gamma\}$ being an orthonormal system (where $\mathbf{B}$ is a vector set), that is,

$$\{\boldsymbol{B}, U\boldsymbol{B}\} \;=\; \delta_U \boldsymbol{I}. \tag{2}$$

Thus, multiply both sides of (2) by $\pi_\omega^{ij}(U)$ and sum over $U \in \Gamma$, yielding

$$\Phi(\omega, i, j) \;\triangleq\; \{\boldsymbol{B}, \sum_{U\in\Gamma} U\pi_\omega^{ij}(U)\boldsymbol{B}\} \;=\; \sum_{U\in\Gamma} \boldsymbol{I}\pi_\omega^{ij}(U)\delta_U. \tag{3}$$

Define

$$\mathcal{U}_\omega^{ij} \;\triangleq\; \frac{\dim(\pi_\omega)}{|\Gamma|} \sum_{U\in\Gamma} U\pi_\omega^{ij}(U). \tag{4}$$

Note that while $\pi_\omega^{ij}$ are scalars, $\mathcal{U}_\omega^{ij}$ are matrices, of the same dimension as $U$. By using (4) in (3), and remembering that $\pi_\omega(\mathcal{I}) = \boldsymbol{I}$ for each $\omega$, one obtains

$$\frac{|\Gamma|}{\dim(\pi_\omega)}\{\boldsymbol{B}, \mathcal{U}_\omega^{ij}\boldsymbol{B}\} \;=\; \boldsymbol{I}\pi_\omega^{ij}(\mathcal{I}) \;=\; \delta_{i-j}\boldsymbol{I}, \tag{5}$$

that is,

$$\{\boldsymbol{B}, \mathcal{U}_{\omega}^{ij}\boldsymbol{B}\} \;=\; \frac{\dim(\pi_{\omega})}{|\Gamma|}\boldsymbol{I}\delta_{i-j}. \tag{6}$$

Note the similarity of (6) with (43) and (42) from [1] (in (42) and (43) indices $i$ and $j$ do not appear because for abelian groups each representation is a scalar).

As in the treatment of abelian groups, (6) is our analysis tool; now we want to find the corresponding orthonormalization tool. Conceptually, dealing with abelian groups does not differ from dealing with nonabelian ones; we thus limit ourselves to stating and proving the algorithm.

A major role in the proof is played by the following property, a generalization of Property 1 from [1], which we give here without proof:

**Property 1** *If matrices* $\mathcal{U}_{\omega}^{ij}$ *are as in (4), then*

$$\begin{aligned} \mathcal{U}_{\omega_1}^{i_1 j_1}\mathcal{U}_{\omega_2}^{i_2 j_2} &= \delta_{j_1 - i_2}\delta_{\omega_1-\omega_2}\mathcal{U}_{\omega_1}^{i_1 j_2}, \\ \left(\mathcal{U}_{\omega}^{ij}\right)^T &= \overline{\mathcal{U}_{\omega}^{ji}}. \end{aligned} \tag{7}$$

Equation (7) tells us that $\mathcal{U}_{\omega}^{ii}$ are projections; therefore it makes sense to speak about a basis of the space $\mathcal{V}_{\omega}^{i}$ associated with $\mathcal{U}_{\omega}^{ii}$. A geometric interpretation of (7) is given later in this section.

To obtain a vector set $\hat{\boldsymbol{B}}$ with $N$ vectors which are orthogonal with respect to the action of $\Gamma$, proceed as follows:

**Algorithm 1**

1. For each $\omega$, find an orthonormal basis of $\mathcal{V}_{\omega}^{1}$ and extract from it $\dim(\pi_{\omega})$ blocks, $\bar{B}_{\omega,1}, \bar{B}_{\omega,2}, \ldots, \bar{B}_{\omega,\dim(\pi_{\omega})}$, each one consisting of $N$ vectors. Of course, the dimension of $\mathcal{V}_{\omega}^{1}$ must be greater or equal than $\dim(\pi_{\omega})N$. (This might seem an unnecessary restriction at this point, but is perfectly reasonable, as is demonstrated shortly).

2. Compute

$$\hat{\boldsymbol{B}} \;=\; \sum_{\omega}\sum_{j=1}^{\dim(\pi_{\omega})}\mathcal{U}_{\omega}^{j1}\bar{B}_{\omega,j}. \tag{8}$$

3. The vector set $\hat{\boldsymbol{B}}$ is orthonormal with respect to the action of $\Gamma$.

The proof is similar to that for the abelian groups which is given in Appendix C of [1].

In the first step of Algorithm 1 we required that the dimension of $\mathcal{V}_{\omega}^{1}$ be at least $\dim(\pi_{\omega})N$. Let us examine this requirement:

1. As proven in the following (see Corollary 1), vector spaces $\mathcal{V}_\omega^j$ have the same dimension (that is assumed to be at least $\dim(\pi_\omega)N$).

2. Therefore, the vector space $\mathcal{V}_\omega$ associated with $\mathcal{P}_\omega = \sum_j \mathcal{U}_\omega^{jj}$, being the direct sum of $\mathcal{V}_\omega^j$, has dimension at least $\dim(\pi_\omega)^2 N$.

3. Finally, vector space $\mathcal{V}$, being direct sum of the vector spaces $\mathcal{V}_\omega$, has dimension at least

$$\sum_\omega \dim(\pi_\omega)^2 N \;=\; N \sum_\omega \dim(\pi_\omega)^2. \tag{9}$$

4. A theorem of group theory states that the sum on the right-hand side of (9) is equal to the cardinality of $\Gamma$ (see below Corollary 4 in Appending A of [1]), that is, the dimension of $\mathcal{V}$ has to be at least

$$\dim(\mathcal{V}) \;\geq\; |\Gamma| N. \tag{10}$$

5. If we apply $\Gamma$ to a vector set $\boldsymbol{B}$ of dimension $N$, we obtain $|\Gamma|N$ vectors. Furthermore, if we want these vectors to be orthogonal to one another (which implies linear independence) condition (10) is obvious.

The first step of Algorithm 1 requires one to choose a set of vectors belonging to $\mathcal{V}_\omega^1$ and orthogonal to one another. Since $\mathcal{U}_\omega^{11}$ is the projection associated with $\mathcal{V}_\omega^1$ one can simply extract some linearly independent columns from $\mathcal{U}_\omega^{11}$ and orthogonalize them. Another way of obtaining such a vector set is given later.

## 2.2 Geometric Interpretation

In this subsection we give a geometric interpretation of the algorithm we just presented. Such an interpretation is instrumental in Section 3 where it is used to produce a parameterization of a basis orthogonal with respect to the action of $\Gamma$.

More precisely, it is shown that with each $\mathcal{U}_\omega^{jj}$ we can associate a vector space $\mathcal{V}_\omega^j$ and that matrices $\mathcal{U}_\omega^{ij}$ act as linear transformations between such spaces. This structure enable us to find a basis of $\mathcal{V}$ such that each $\mathcal{U}_\omega^{ij}$ has a simple form.

### 2.2.1 Structure of $\mathcal{V}$

We already observed that $\mathcal{U}_\omega^{ii}$ are orthogonal projections. Now we want to prove that matrix $\mathcal{U}_\omega^{ij}$ is a linear transformation mapping $\mathcal{V}_\omega^j$ into $\mathcal{V}_\omega^i$; more precisely, we have the following property, whose proof can be found in Appendix A:

**Property 2** *$\mathcal{U}_\omega^{ij}$ is an invertible linear transformation between $\mathcal{V}_\omega^j$ and $\mathcal{V}_\omega^i$, that is, $\mathcal{V}_\omega^i = \mathcal{U}_\omega^{ij}\mathcal{V}_\omega^j$ and* $\ker\left(\mathcal{U}_\omega^{ij}\right) \cap \mathcal{V}_\omega^j = \{0\}$.

A corollary easily follows:

**Corollary 1** *Vector spaces $\mathcal{V}_\omega^j$ and $\mathcal{V}_\omega^i$ have the same dimension.*

Note that $\mathcal{U}_\omega^{ij}$ is *not* invertible if considered as a linear transformation from $\mathcal{V}$ in itself, but it becomes invertible when thought of as a map from $\mathcal{V}_\omega^j$ to $\mathcal{V}_\omega^i$.

It is worth observing that there is no linear transformation (in the set of $\mathcal{U}_\omega^{ij}$) linking two spaces relative to two different representations (that is, having different $\omega$). This situation is depicted in Figure 1 where each space $\mathcal{V}_\omega^j$, associated with projection $\mathcal{U}_\omega^{jj}$, is represented by a box labeled by the projection itself and the boxes are connected by branches labeled by the name of the matrix $\mathcal{U}_\omega^{ij}$ that maps one space in another. Not all the branches are shown in order not to clog the figure.

The sum of all the projections corresponding to the same representation is

$$\mathcal{P}_\omega \triangleq \sum_j \mathcal{U}_\omega^{jj}, \tag{11}$$

and is still a projection because $\mathcal{U}_\omega^{jj}$ are orthogonal to one another. The corresponding space is represented in Figure 1 as a line encircling the spaces relative to $\mathcal{U}_\omega^{jj}$ and labeled with $\mathcal{P}_\omega$. The collection of all spaces $\mathcal{V}_\omega$ forms the complete vector space $\mathcal{V}$.

It is interesting to observe that the spaces relative to $\mathcal{U}_\omega^{jj}$ in Figure 1 can be interpreted as "clans" (represented by projections $\mathcal{P}_\omega$) in which every member can go into any other member of the same clan, but not into a member of a different one.

We will see later that such a clan structure imposes certain conditions on the spaces belonging to the same clan, but not on the spaces from different clans.

### 2.2.2 A Particular Basis

This geometric interpretation is useful because it allows us to choose a suitable basis for $\mathcal{V}$ such that matrices $\mathcal{U}_\omega^{ij}$ have a simple form. (The consequences of this are exploited in Section 3.)

Since $\mathcal{V}$ is a direct sum of the spaces $\mathcal{V}_\omega^j$ (associated with $\mathcal{U}_\omega^{jj}$), a basis of $\mathcal{V}$ can be obtained by a direct sum of bases of $\mathcal{V}_\omega^j$. The resulting structure of a vector from $\mathcal{V}$ is depicted in Figure 2(a) where it is displayed as a sequence of blocks, each block being associated with a $\mathcal{U}_\omega^{jj}$. The blocks relative to the same $\omega$ can be thought of as making a "macro-block" relative to the space $\mathcal{V}_\omega$ (associated with $\mathcal{P}_\omega$). The remainder of the figure will be explained later.

We proceed in the following way:

- First, we find a basis for the space $\mathcal{V}_\omega^1$.
- Next, such a basis is modified to obtain the bases for each $\mathcal{V}_\omega^j$.
- Finally, by repeating the previous two steps for each $\omega$, we have the basis for $\mathcal{V}$.

### 2.2.3 Finding a Basis for $\mathcal{V}_\omega^1$

Consider the matrix $\mathcal{U}_\omega^{11}$ and choose rank $\left(\mathcal{U}_\omega^{11}\right)$ of its linearly independent columns as a basis for the associated vector space $\mathcal{V}_\omega^1$. Then, orthogonalize them (using the Gram-Schmidt orthogonalization procedure, for example). Such vectors form an orthonormal basis for $\mathcal{V}_\omega^1$.

With respect to such a basis, the linear transformation $\mathcal{U}_\omega^{11}$, restricted to $\mathcal{V}_\omega^1$, is represented by an identity matrix because each vector of the basis maps in itself with the action of $\mathcal{U}_\omega^{11}$. Moreover, if the basis of $\mathcal{V}$ is chosen as direct sum of bases of $\mathcal{V}_\omega^j$, matrix $\mathcal{U}_\omega^{11}$, with respect to such a basis, is a diagonal matrix having 1's in correspondence to the block relative to $\mathcal{V}_\omega^1$ and zero otherwise; therefore it is a pseudo-identity[1].

### 2.2.4 Finding a Basis for each $\mathcal{V}_\omega^j$

To modify the basis for $\mathcal{V}_\omega^1$ to obtain bases for $\mathcal{V}_\omega^j$, remember that $\mathcal{U}_\omega^{j1}$ is an invertible transformations between $\mathcal{V}_\omega^1$ and $\mathcal{V}_\omega^j$, and that vector spaces $\mathcal{V}_\omega^1$ and $\mathcal{V}_\omega^j$ have the same dimension. Therefore, a basis for $\mathcal{V}_\omega^j$ can be obtained by applying $\mathcal{U}_\omega^{j1}$ to a basis of $\mathcal{V}_\omega^1$.

By transforming a vector $\mathcal{U}_\omega^{j1}\boldsymbol{v}$, $\boldsymbol{v} \in \mathcal{V}_\omega^1$, with $\mathcal{U}_\omega^{jj}$ we obtain

$$\mathcal{U}_\omega^{jj}\mathcal{U}_\omega^{j1}\boldsymbol{v} \;=\; \mathcal{U}_\omega^{j1}\boldsymbol{v}, \tag{12}$$

because of (7). Equation (12) says that, with this particular choice of a basis, $\mathcal{U}_\omega^{jj}$ is a pseudo-identity as well.

We see that by using these bases, $\mathcal{P}_\omega$, being the direct sum of $\mathcal{U}_\omega^{jj}$, is still represented as a diagonal matrix with 1's in the blocks corresponding to $\omega$ and zero otherwise (that is, $\mathcal{P}_\omega$ is a pseudo-identity as well).

[1] By pseudo-identity, we denote diagonal matrices having only zeros and ones on the main diagonal.

A pictorial representation of the effect both of $\mathcal{P}_\omega$ and $\mathcal{U}_\omega^{jj}$ is given in Figures 2(b) and 2(c). There, the action of $\mathcal{P}_\omega$ on a vector, depicted as a row of "blocks", is seen as a sieve, passing all $\mathcal{U}_\omega^{jj}$ and stopping $\mathcal{U}_{\omega_1}^{jj}$, for $\omega_1 \neq \omega$. The interpretation of $\mathcal{U}_\omega^{jj}$ is similar, only with a thinner sieve.

To understand Figure 2(d) depicting the action of $\mathcal{U}_\omega^{ij}$, note that if $\boldsymbol{v} \in \mathcal{V}_\omega^1$, then vector $\mathcal{U}_\omega^{j1}\boldsymbol{v} \in \mathcal{V}_\omega^j$ will have, with respect to the basis $\mathcal{U}_\omega^{j1}\mathcal{B}$, the same components that $\boldsymbol{v}$ has with respect to the basis of $\mathcal{B}$. Therefore, the effect of $\mathcal{U}_\omega^{j1}$ can be seen as a movement of the block corresponding to $\mathcal{U}_\omega^{11}$ to the position corresponding to $\mathcal{U}_\omega^{jj}$. In Figure 2(c) this is depicted as a sieve which causes block movement using a bent output channel.

### 2.2.5 Geometric Interpretation of Orthogonality Conditions

With such a basis choice we can give an interesting interpretation to (6). Let us suppose, for simplicity, that the vector set $\boldsymbol{B}$ is actually a single vector $\boldsymbol{b}$. Then, (6) becomes

$$\langle \boldsymbol{b}, \mathcal{U}_\omega^{ij}\boldsymbol{b}\rangle = \frac{\dim(\pi_\omega)}{|\Gamma|}\delta_{i-j}. \tag{13}$$

Let us interpret (13) in the spirit of Figure 2 with the help of Figure 3. Figure 3 shows that the action of $\mathcal{U}_\omega^{ij}$ is to put block $\boldsymbol{b}_2$ into the position of block $\boldsymbol{b}_1$ and zero otherwise. When computing the scalar product between $\boldsymbol{b}$ and $\mathcal{U}_\omega^{ij}\boldsymbol{b}$ it is clear that one obtains something that can be loosely called "the scalar product between blocks $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$." Therefore, (13) is a condition between two blocks belonging to the same macro-block; note that there is no constraint on the blocks of different macro-blocks.

These reasonings can be summarized as follows:

> Conditions (6), for the orthogonality of $\boldsymbol{b}$ with respect to the action of $\Gamma$, mean that, with respect to our canonical basis, the blocks of $\boldsymbol{b}$ belonging to the same macro-block have to be of unit norm and orthogonal to one another.

The same reasoning can also be carried out more formally by multiplying $\mathcal{U}_\omega^{ij}$ in (13) by $\mathcal{U}_\omega^{ii}$ on the left and by $\mathcal{U}_\omega^{jj}$ on the right. It is possible to do so because of (7). Then, $\mathcal{U}_\omega^{ii}$ can be brought on the left side of the scalar product (because it is self-adjoint) and an interpretation analogous to Figure 3 can be given.

Note that for abelian groups there is only one block for each macro-block; because of this, the condition of cross-orthogonality between blocks disappears and only the condition on the unitary norm remains, as previously seen.

### 2.2.6 Parameterization of Vectors Satisfying (13)

The geometric interpretation can be used to express a vector satisfying (13) as a function of certain free parameters. This can be useful, for example, while designing $\boldsymbol{b}$ using an optimization technique (as will be seen in Section 3) because the resulting problem is unconstrained.

Figure 4 shows how to proceed. Vector $\boldsymbol{b}$ is decomposed into macro-blocks and the blocks of each macro-block are organized as the columns of a matrix $\boldsymbol{A}_{\omega_k}$. Orthogonality condition between blocks implies that the columns of matrix $\boldsymbol{A}_{\omega_k}$ are orthogonal to one another, that is

$$\boldsymbol{A}_{\omega_k}^T \boldsymbol{A}_{\omega_k} = \boldsymbol{I}. \tag{14}$$

Note that there are no constraints between matrices with different $\omega$.

We can summarize these reasonings as follows:

> Each vector $\boldsymbol{b}$ orthogonal with respect to the action of $\Gamma$ can be constructed by choosing a set of orthogonal matrices $\boldsymbol{A}_{\omega_k}$ and using their columns as blocks of $\boldsymbol{b}$.

The importance of the above is that an orthogonal matrix (even if not square) can be parameterized using Givens' rotations [2]. This procedure yields the desired description of $\boldsymbol{b}$ as a function of free parameters (the angles of Givens' rotations).

Figure 5 shows the reconstruction of $\boldsymbol{b}$ using the Givens rotations $\alpha_{kn}$. Figure 5(a) shows how each block is associated with a set of rotations $\alpha_{kn}$ that are used to construct the matrix $\boldsymbol{A}_{\omega_k}$ (Figure 5(b)). The columns of such a matrix are subsequently used in Figure 5(c) as blocks of $\boldsymbol{b}$.

## 2.3 An Algorithm to Choose the Orthonormal Basis

As explained in Section 2.1, we need, for each $\omega$, a set of vectors $\bar{B}_{\omega,1}$, $\bar{B}_{\omega,2}, \ldots, \bar{B}_{\omega,\dim(\pi_\omega)}$ belonging to $\mathcal{V}_\omega^1$ and orthogonal to one another. While it is possible to choose such vectors from an orthonormal basis of $\mathcal{V}_\omega^1$, it would be interesting if we could obtain them from the starting vector set $\boldsymbol{B}$.

Remember that for abelian groups there is a way to modify $\boldsymbol{B}$ to obtain a set orthonormal with respect to the action of $\Gamma$. In this section we present an analogous method that works for nonabelian groups.

Observe how vector sets $\bar{B}_{\omega,j}$ are used: They are multiplied by $\mathcal{U}_\omega^{1j}$ and summed in (8). Each term $\mathcal{U}_\omega^{1j}\bar{B}_{\omega,j}$ in

(8) is the projection of $\hat{\boldsymbol{B}}$ on $\mathcal{V}_\omega^j$; indeed

$$\mathcal{U}_\omega^{jj}\hat{\boldsymbol{B}} \;=\; \mathcal{U}_\omega^{jj}\sum_\omega \sum_{j=1}^{\dim(\pi_\omega)} \mathcal{U}_\omega^{j1}\bar{B}_{\omega,j} \;=\; \mathcal{U}_\omega^{jj}\mathcal{U}_\omega^{j1}\bar{B}_{\omega,j} \;=\; \mathcal{U}_\omega^{j1}\bar{B}_{\omega,j}, \tag{15}$$

because of Property 1. If the canonical basis described previously is used, $\mathcal{U}_\omega^{j1}\bar{B}_{\omega,j}$ is the block of $\hat{\boldsymbol{B}}$ corresponding to $\mathcal{V}_\omega^j$. If $\boldsymbol{B}$ is not orthogonal with respect to the action of $\Gamma$ it is because its blocks are not orthogonal to one another. A remedy is to project $\boldsymbol{B}$ onto $\mathcal{V}_\omega^j$, map the projections into $\mathcal{V}_\omega^1$, orthogonalize the obtained vectors and map them back into their spaces $\mathcal{V}_\omega^j$. This, obviously, yields a set of vectors suitable for use in Algorithm 1. Let us summarize these observations in the form of an algorithm:

**Algorithm 2**

1. For each $\omega$

   1.1. For $j = 1, \ldots, \dim(\pi_\omega)$

      1.1.1. Project $\boldsymbol{B}$ on $\mathcal{V}_\omega^j$ in order to obtain $\boldsymbol{B}_j$.

      1.1.2. Apply $\mathcal{U}_\omega^{1j}$ to $\boldsymbol{B}_j$ to obtain $\boldsymbol{B}_j^1 \triangleq \mathcal{U}_\omega^{1j}\boldsymbol{B}_j \in \mathcal{V}_\omega^1$.

   1.2. Concatenate vector sets $\boldsymbol{B}_j^1$ to obtain a larger vector set $\boldsymbol{B}^+$. Note that the orthogonality condition can be expressed as $\{\boldsymbol{B}^+, \boldsymbol{B}^+\} = \boldsymbol{I}$.

   1.3. In general, matrix $\boldsymbol{A} \triangleq \{\boldsymbol{B}^+, \boldsymbol{B}^+\}$ is not an identity; it is always a positive semi-definite matrix. Assume that $\boldsymbol{A}$ is invertible and decompose $\boldsymbol{A}$ with the singular value decomposition as $\boldsymbol{A} = \boldsymbol{O}^T\boldsymbol{S}^2\boldsymbol{O}$, with $\boldsymbol{S}$ having only nonnegative values on the main diagonal.

   1.4. Define $\hat{\boldsymbol{B}^+} \triangleq \boldsymbol{B}^+\boldsymbol{O}^T\boldsymbol{S}^{-1}\boldsymbol{O}$. (The definition of the product of a vector set with a matrix can be found in Appendix B.) Then, $\hat{\boldsymbol{B}^+}$ satisfies the orthogonality condition; indeed,

$$\begin{aligned}
\{\hat{\boldsymbol{B}^+}, \hat{\boldsymbol{B}^+}\} &= \{\boldsymbol{B}^+\boldsymbol{O}^T\boldsymbol{S}^{-1}\boldsymbol{O}, \boldsymbol{B}^+\boldsymbol{O}^T\boldsymbol{S}^{-1}\boldsymbol{O}\}, \\
&= \left(\boldsymbol{O}^T\boldsymbol{S}^{-1}\boldsymbol{O}\right)^T \{\boldsymbol{B}^+, \boldsymbol{B}^+\}\boldsymbol{O}^T\boldsymbol{S}^{-1}\boldsymbol{O}, \\
&= \left(\boldsymbol{O}^T\boldsymbol{S}^{-1}\boldsymbol{O}\right)^T \boldsymbol{O}^T\boldsymbol{S}^2\boldsymbol{O}\left(\boldsymbol{O}^T\boldsymbol{S}^{-1}\boldsymbol{O}\right) \;=\; \boldsymbol{I}.
\end{aligned} \tag{16}$$

   1.5. Disassemble $\hat{\boldsymbol{B}^+}$ in order to obtain blocks $\bar{B}_{\omega,j}$.

Note that in Algorithm 2 the columns of $\boldsymbol{A}$ are orthogonalized using the singular value decomposition and not the more common Gram-Schmidt orthogonalization procedure. The use of the singular value decomposition gives to Algorithm 2 the following interesting property, whose proof can be found in Appendix A:

**Property 3** *When vectors are chosen as described in Algorithm 2, vector set $\hat{\boldsymbol{B}}$ is the vector set orthogonal with respect to the action of $\Gamma$ having the minimum distance from $\boldsymbol{B}$.*

Note that Property 3 can be seen as the generalization of Property 3 from [1], which will be exploited in Section 3 to help our filter design.

# 3 Filter Design

The theory presented in the previous sections describes how to find a set orthonormal with respect to a transformation group $\Gamma$. In particular, we are searching for bases with local orthogonal structure [3], [4].

Local orthogonal bases extend the well-known modulated lapped transforms (also known as local trigonometric bases) [5] to a more general setting. In [3], the impulse responses of the filters constituting the filter bank are expressed as the columns of the matrix (final basis)

$$\mathcal{W}\mathcal{K}\boldsymbol{G}, \tag{17}$$

where $\boldsymbol{G}$ is a unitary matrix (starting basis), and $\mathcal{W}$ and $\mathcal{K}$ are two matrices (windowing and symmetry reduction) which depend on the filter bank structure and whose exact form does not matter for the purposes of this work. Because of this, in the following we simply use the matrix $\mathcal{L} \stackrel{\triangle}{=} \mathcal{W}\mathcal{K}$. In [4] the design of matrix $\mathcal{W}$ is discussed, while $\boldsymbol{G}$ is left undetermined; its design is the goal of this section.

In the introduction we suggested that a filter bank invariant under the action of a given group can be useful to analyze an image and to find its local symmetries and regularities. The technique in [3] allows us to design a multidimensional filter bank starting from an orthonormal basis. If such a basis is designed to be invariant with respect to a group action we can obtain a set of filters that can be used for this purpose.

## 3.1 Problem Statement

Therefore, we require $\boldsymbol{G}$ to be a matrix whose columns are obtained by applying a transformation group $\Gamma$ to a single vector $\boldsymbol{g}$, that is,

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g} & U_1\boldsymbol{g} & U_2\boldsymbol{g} & \cdots \end{bmatrix}, \tag{18}$$

with $U_n \in \Gamma$. More generally we could ask that the columns of $\boldsymbol{G}$ be obtained with the action of $\Gamma$ on two or more vectors, for example

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & U_1\boldsymbol{g}_1 & U_1\boldsymbol{g}_2 & \cdots \end{bmatrix}. \tag{19}$$

For the sake of simplicity, we limit ourselves to having only one vector. Bearing this in mind, the filters of the designed filter bank have impulse responses $\mathcal{L}\boldsymbol{g}$, $\mathcal{L}U_1\boldsymbol{g}$ ,$\mathcal{L}U_2\boldsymbol{g}, \ldots$, with $U_n \in \Gamma$.

### 3.2 Cost Function

It is common practice to design filters optimizing some measure of frequential and/or temporal error. We could, for example, specify in which region of the time-frequency plane should our filter mostly reside. That way we impose a certain time-frequency localization on the filter. Then, we can express the power in time and frequency domains, respectively, as

$$\begin{aligned} P_T &= \sum_n \rho[n]\boldsymbol{g}^2[n] \;=\; \boldsymbol{g}^T\mathbf{C}_T\boldsymbol{g}, \\ P_F &= \boldsymbol{g}^T\left(\int_\omega \rho(\omega)\mathbf{W}(\omega)\mathbf{W}^T(\omega)d\omega\right)\boldsymbol{g} \;=\; \boldsymbol{g}^T\mathbf{C}_F\boldsymbol{g}, \end{aligned} \tag{20}$$

where $\boldsymbol{g}$ is our filter, $\rho(\omega)$ and $\rho[n]$ are weighting functions describing the prescribed time-frequency region, $\mathbf{W}(\omega)$ is a column vector containing $\exp(j\omega n)$ and $\mathbf{C}_T$ and $\mathbf{C}_F$ are time and frequency costs, respectively. By summing $P_T$ and $P_F$ we see that our objective function is a a quadratic function of each filter $\boldsymbol{g}$, that is, the cost function has the form (we also add $\mathcal{L}U$)

$$\sum_{U\in\Gamma}(\mathcal{L}U\boldsymbol{g})^T\boldsymbol{C}_U\mathcal{L}U\boldsymbol{g}. \tag{21}$$

Note that (21) could become a weighted sum by multiplying each term by a real number. Such a number can successively be absorbed in the corresponding cost matrix, as we assume in the following.

Let us modify (21) to obtain a simpler expression:

$$\begin{aligned} \sum_{U\in\Gamma}(\mathcal{L}U\boldsymbol{g})^T\boldsymbol{C}_U\mathcal{L}U\boldsymbol{g} &= \sum_{U\in\Gamma}\boldsymbol{g}^TU^T\mathcal{L}^T\boldsymbol{C}_U\mathcal{L}U\boldsymbol{g}, \\ &= \boldsymbol{g}^T\left(\sum_{U\in\Gamma}U^T\mathcal{L}^T\boldsymbol{C}_U\mathcal{L}U\right)\boldsymbol{g}, \\ &= \boldsymbol{g}^T\boldsymbol{C}\boldsymbol{g}, \end{aligned} \tag{22}$$

where

$$\boldsymbol{C} \triangleq \sum_{U\in\Gamma}U^T\mathcal{L}^T\boldsymbol{C}_U\mathcal{L}U. \tag{23}$$

Equation (22) shows that the global cost function (21) can be expressed as a quadratic cost of the single vector $\boldsymbol{g}$, yielding a simpler problem.

### 3.3 True Cost Function Minimization

As seen in the previous sections, the vector $\boldsymbol{g}$ is not free since the demand that it be orthogonal with respect to the action of group $\Gamma$ gives to $\boldsymbol{g}$ a particular structure. The parameterization developed in Section 2, that allows us to obtain $\boldsymbol{g}$ as a function of certain free parameters that can be interpreted as certain Givens' rotations, will be useful.

We now have a simple problem of unconstrained optimization with a quadratic cost function that can be easily solved with numerical methods.

### 3.4 A Simpler Approach

Often the use of a quadratic cost is motivated more by theoretical convenience than by a real cost with a quadratic characteristic. Bearing this in mind, a solution that might not achieve the minimum cost, but is close to the optimal solution can be interesting if, for example, it is simpler to compute. Indeed, to obtain the vector $\boldsymbol{g}$ from Givens' rotations is theoretically simple, but is computationally intensive, and when included in an optimization loop, can give rise to long design times. Although filter design is usually made off-line, that is, without severe fast computation requirements, a faster design procedure can allow for trying several types of filters. Therefore, a suboptimal, but faster solution can be interesting. To this end we can exploit Property 3 stating that, if we choose our vector in the particular way described in Section 2, the resulting vector has minimum distance from the original one.

The idea is to minimize function (22) with $\|\boldsymbol{g}\| = 1$ as the only constraint. Such a condition is necessary because otherwise we could achieve arbitrarily small values by simply scaling $\boldsymbol{g}$. This problem has a well-known solution: $\boldsymbol{g}$ is the eigenvector of $\boldsymbol{C}$ having the minimum eigenvalue.

Now, if we apply Algorithm 2 to such a $\boldsymbol{g}$, we obtain the vector $\hat{\boldsymbol{g}}$ orthogonal with respect to the action of $\Gamma$ and having a minimum distance from $\boldsymbol{g}$. Since both vectors are constrained to be of unit norm, the minimum distance is equivalent to the minimum angle between the two vectors, that is,

$$\|\boldsymbol{g} - \hat{\boldsymbol{g}}\| \;=\; \langle \boldsymbol{g} - \hat{\boldsymbol{g}}, \boldsymbol{g} - \hat{\boldsymbol{g}} \rangle \;=\; \|\boldsymbol{g}\| + \|\hat{\boldsymbol{g}}\| - 2\langle \boldsymbol{g}, \hat{\boldsymbol{g}} \rangle \;=\; 2 - 2\cos(\alpha), \tag{24}$$

where $\alpha = \arccos(\langle \boldsymbol{g}, \hat{\boldsymbol{g}} \rangle)$ is the angle between the two vectors. It is worth observing that we can always impose $\cos(\alpha) = \langle \boldsymbol{g}, \hat{\boldsymbol{g}} \rangle \geq 0$ because we can substitute $\hat{\boldsymbol{g}}$ with $-\hat{\boldsymbol{g}}$ without changing the cost function. The effect on the filter coefficients is simply a change of sign. To have a vector with the minimum distance from the (unconstrained) optimal one is acceptable, but we would like to know if it is necessarily the optimum (with the constraint for it to be orthogonal with respect to $\Gamma$). Although we are not guaranteed that $\hat{\boldsymbol{g}}$ is the optimum, we can obtain an estimate of how much $\hat{\boldsymbol{g}}$ is far from the optimum.

To understand what can happen, remember that we assumed $\boldsymbol{C}$ to be symmetric. Because of this, it can be diagonalized and by rescaling we can set the minimum eigenvalue equal to 1. Therefore, we can assume $\boldsymbol{C}$ to be diagonal with the eigenvalues ordered along the main diagonal as

$$\boldsymbol{C} = \begin{bmatrix} 1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_N \end{bmatrix}, \tag{25}$$

with $\mu_1 = 1 \leq \mu_2 \leq \cdots \leq \mu_N$. Then, in the unconstrained optimization, we obtain $\boldsymbol{g} = [1, 0, \ldots, 0]^T$. Since its normalized $\hat{\boldsymbol{g}}$ version is such that $\langle \boldsymbol{g}, \hat{\boldsymbol{g}} \rangle = \cos(\alpha)$ and $||\hat{\boldsymbol{g}}|| = 1$, the following is true:

$$\hat{\boldsymbol{g}} = [\cos(\alpha), \sin(\alpha)\boldsymbol{v}]^T, \tag{26}$$

where $\boldsymbol{v}$ is a vector with $||\boldsymbol{v}|| = 1$. The value that the objective function (22) attains on vector (26) is

$$\cos^2(\alpha) + \sin^2(\alpha)\boldsymbol{v}^T \boldsymbol{C} \boldsymbol{v}. \tag{27}$$

Since $||\boldsymbol{v}|| = 1$, the value of the second term in (27) lies between $\sin^2(\alpha)\mu_2$ and $\sin^2(\alpha)\mu_N$. From (27) it is clear that, if $\alpha$ is fixed, the lesser the value of $\boldsymbol{v}^T \boldsymbol{C} \boldsymbol{v}$, the lesser the value of (27) (since $\sin^2(\alpha) \geq 0$). Since $\boldsymbol{v}$ is independent from $\alpha$ we can choose it equal to $[0, 1, 0, \ldots]^T$ and the worst possible value of (27) becomes equal to

$$\cos^2(\alpha) + \sin^2(\alpha)\mu_2. \tag{28}$$

We know that $\alpha$ is the minimum possible angle between $\boldsymbol{g}$ and $\hat{\boldsymbol{g}}$. We would like to say that (28) is the minimum cost that a vector having an angle greater or equal to $\alpha$ with $\boldsymbol{g}$ can achieve. Can we choose $\alpha_1 > \alpha$ in order to decrease the value of (28)? It is easy to see that it is not possible. Although easy to prove it formally, a geometric interpretation is clearer. Equation (28) is the distance from the origin of a point lying on an ellipse centered in the origin and having the axis of length 1 and $\mu_2 > 1$, respectively. The situation is depicted in Figure 6 where the shadowed zone is the forbidden zone. From Figure 6 it is clear that it is not possible to choose a greater $\alpha_1$ and obtain a lesser value of (28). Therefore, the minimum value that the cost function can achieve on the set of the vectors orthogonal with respect to the action of $\Gamma$ is (28).

Since the value of $\alpha$ can be estimated by the scalar product $\langle \boldsymbol{g}, \hat{\boldsymbol{g}} \rangle$ and the value of $\mu_2$ can be easily obtained by matrix $\boldsymbol{C}$, (28) can be used to get a rough idea if we are close or not to the minimum. Note that the estimate (28) can be pessimistic although we might have reached the true minimum.

# 4 Design of Filters on Groups of Modulations and Rotations

As an example of using the noncommutative groups, we study the case that triggered this work, that is, the group of rotations and modulations (translations in frequency). We do this in the following order: determine the frequency translations group $\mathcal{T}$, determine the frequency rotation group $\mathcal{R}$, put $\mathcal{T}$ and $\mathcal{R}$ together in order to obtain $\Gamma$, find the irreducible representations of $\Gamma$ and finally, design the prototype filter and the corresponding orthonormal set obtained by translations and rotations.

## 4.1 Determining the Group of Modulations

We start by determining the modulation group. Let $s(\mathbf{n})$, $\mathbf{n} \in \mathbb{Z}^2$, be a two-dimensional signal and define its Fourier transform as

$$S(\boldsymbol{\omega}) \stackrel{\triangle}{=} \sum_{\mathbf{n} \in \mathbb{Z}^2} \exp(-j2\pi \boldsymbol{\omega}^T \mathbf{n}) s(\mathbf{n}). \tag{29}$$

Note that in (29) the frequency variable $\boldsymbol{\omega} = [\omega_1, \omega_2]^T$ has been scaled in such a way that $S(\boldsymbol{\omega})$ is periodic[2] on $\mathbb{Z}^2$. The Fourier transform of the modulated version of $s(\mathbf{n})$, that is, of $\exp(-j2\pi {\boldsymbol{\omega}_0}^T \mathbf{n}) s(\mathbf{n})$, is $S(\boldsymbol{\omega} + \boldsymbol{\omega}_0)$. Since translations in frequency are easier to handle than modulations in time, we work in the frequency domain. As the frequency translation group we use the lattice generated by a rational matrix $\mathbf{Q}$

$$\Lambda(\mathbf{Q}) \stackrel{\triangle}{=} \{\mathbf{x} \mid \mathbf{x} = \mathbf{Q}\mathbf{m}, \mathbf{m} \in \mathbb{Z}^2\}. \tag{30}$$

Group $\Lambda(\mathbf{Q})$ poses a technical problem: the theory developed in the preceding sections works for finite groups, but the cardinality of $\Lambda(\mathbf{Q})$ is infinite. This cannot be avoided by choosing another group because if $\boldsymbol{\omega}_0 \neq \mathbf{0}$ belongs to $\mathcal{T}$, every integer multiple of $\boldsymbol{\omega}_0$ belongs to $\mathcal{T}$ as well, giving rise to an infinite group. However, here we can exploit the periodicity of $S(\boldsymbol{\omega})$. If the condition $\Lambda(\mathbf{Q}) \supset \mathbb{Z}^2$ is imposed, one can choose for $\mathcal{T}$ the quotient group $\Lambda(\mathbf{Q})/\mathbb{Z}^2$, that is, $\mathcal{T} = \{\mathbf{m} \mid \mathbf{m} = \mathbf{k}(\mod \mathbb{Z}^2), \mathbf{k} \in \Lambda(\mathbf{Q})\}$. It is possible to prove that the cardinality of $\mathcal{T}$ is $1/|\det(\mathbf{Q})| < \infty$. In this example we use $\mathbf{Q} = \mathrm{diag}(1/16, 1/16)$.

## 4.2 Determining the Group of Rotations

Let us now try to determine the rotation group. As a first attempt, one could define the rotation of the signal $s(\mathbf{n})$ as $s(\mathbf{R}\mathbf{n})$, where $\mathbf{R}$ is one of Givens' rotation matrices. However, since the signal $s(\mathbf{n})$ is defined on $\mathbb{Z}^2$, matrix $\mathbf{R}$ must be an integer matrix for $s(\mathbf{R}\mathbf{n})$ to make sense for every $\mathbf{n}$. Therefore, we need an "approximate rotation."

[2] This convention is not commonly used in signal processing where $S(\omega)$ is periodic on $2\pi\mathbb{Z}^2$. However, it simplifies many later derivations.

Observe that if in a Givens' rotations matrix one poses $\alpha_0 = 2\pi/k$ then matrix $\mathbf{R}$ satisfies the following:

$$\begin{cases} \mathbf{R}^k & = \mathbf{I} \\ \mathbf{R}^n & \neq \mathbf{I}, \qquad 0 < n < k. \end{cases} \tag{31}$$

Our definition of approximate rotation is inspired by (31) and we search for an integer matrix $\mathbf{R}$ satisfying it. Note that (31) implies that all the eigenvalues of $\mathbf{R}$ are $k$th roots of unity and this agrees with the idea that $\mathbf{R}$ should represent a rotation. Unlike when $\mathbf{R}$ has real elements, it is not trivial to find an integer matrix $\mathbf{R}$ satisfying (31) for a given $k$. Indeed, the following negative result holds: There is no $2 \times 2$ integer matrix $\mathbf{R}$ satisfying (31) if $k = 5$ or $k > 6$. In the following we use $k = 6$.

To find $\mathbf{R}$, observe that the eigenvalues of $\mathbf{R}$ must be $\exp(\pm j\pi/3)$ implying that

$$\det(\mathbf{R} - \mathbf{I}\lambda) \;=\; (\lambda - e^{-j\pi/3})(\lambda - e^{+j\pi/3}) \;=\; \lambda^2 - \lambda + 1 \;=\; \lambda^2 - \mathrm{Tr}(\mathbf{R})\lambda + \det(\mathbf{R}). \tag{32}$$

By calling $r_{ij}$ the generic element of $\mathbf{R}$, from (32) we obtain

$$\begin{aligned} \det(\mathbf{R}) &= r_{11}r_{22} - r_{12}r_{21} = 1 \\ \mathrm{Tr}(\mathbf{R}) &= r_{11} + r_{22} = 1. \end{aligned} \tag{33}$$

One could choose, for example, the matrix

$$\mathbf{R} \;=\; \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \tag{34}$$

for which, $\mathbf{R}^6 = \mathbf{I}$ and $\mathbf{R}^k \neq \mathbf{I}$ for $0 < k < 6$.

Since we decided to handle modulations in the frequency domain, we need to do the same for rotations. The Fourier transform of $s(\mathbf{Rn})$ is $S(\mathbf{R}^{-T}\omega)$, and thus in what follows we use $\mathbb{R}$ instead of $\mathbf{R}^{-T}$ since it is necessary to write powers of $\mathbf{R}^{-T}$. If $\mathbf{R}$ is as in (34), then

$$\mathbb{R} \;=\; \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}. \tag{35}$$

Note that $\mathbb{R}^6 = \mathbf{I}$ and that the set $\mathcal{R} = \{\mathbf{I}, \mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^5\}$ is a group.

### 4.3 Determining the Group of Modulations and Rotations Acting on a Point

Let us now combine modulations and rotations. We first find a condition so that by applying a rotation $\mathbb{R}$ to $\Lambda(\mathbf{Q})$ one must obtain $\Lambda(\mathbf{Q})$ again. If $\Lambda(\mathbf{Q})$ is not changed by rotation $\mathbb{R}$, it is not changed by multiple rotations $\mathbb{R}^k$, that is, $\Lambda(\mathbf{Q})$ is invariant with respect to the action of $\mathcal{R}$. This can be written as: for every $\mathbf{m} \in \mathbb{Z}^2$, $\mathbb{R}\mathbf{Qm} \in \Lambda(\mathbf{Q})$ needs to be satisfied, or, in other words, $\Lambda(\mathbb{R}\mathbf{Q}) \subseteq \Lambda(\mathbf{Q})$, which is equivalent to requiring the existence of an integer matrix $\mathbf{N}$ such that $\mathbb{R}\mathbf{Q} = \mathbf{QN}$, or, equivalently, $\mathbf{N} = \mathbf{Q^{-1}}\mathbb{R}\mathbf{Q} \in \mathbb{Z}^{2\times 2}$. This, in turn implies that

$$\Lambda(\mathbb{R}\mathbf{Q}) \;=\; \Lambda(\mathbf{QN}) \;=\; \Lambda(\mathbf{Q})\,. \tag{36}$$

Under the above assumptions, the smallest group $\Gamma$ containing both our frequency translation group $\mathcal{T}$ and the modulation group $\mathcal{R}$ is the group of affine transformations mapping $\boldsymbol{\omega}$ into

$$\hat{\boldsymbol{\omega}} \;=\; \mathbb{R}^{\mathrm{a}}\boldsymbol{\omega} + \mathbf{m}, \qquad \mathrm{a} \in \mathbb{Z}, \quad \mathbf{m} \in \mathcal{T}. \tag{37}$$

Equation (37) can be written in a more convenient form as

$$\begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbb{R}^{\mathrm{a}} & \mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ 1 \end{bmatrix} \qquad a \in \mathbb{Z}, \quad \mathbf{m} \in \mathcal{T}. \tag{38}$$

Therefore, a generic element of group $\Gamma$ has the form

$$U(\mathbf{m}, a) \;=\; \begin{bmatrix} \mathbb{R}^{\mathrm{a}} & \mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{39}$$

Using (39) we give the composition law of $\Gamma$ and the form of the inverse

$$\begin{aligned} U(\mathbf{m}_1, a_1)U(\mathbf{m}_2, a_2) \;&=\; \begin{bmatrix} \mathbb{R}^{\mathrm{a}_1} & \mathbf{m}_1 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbb{R}^{\mathrm{a}_2} & \mathbf{m}_2 \\ \mathbf{0} & 1 \end{bmatrix} = \\ &=\; \begin{bmatrix} \mathbb{R}^{\mathrm{a}_1+\mathrm{a}_2} & \mathbb{R}^{\mathrm{a}_1}\mathbf{m}_2 + \mathbf{m}_1 \\ \mathbf{0} & 1 \end{bmatrix} \;=\; U(\mathbb{R}^{\mathrm{a}_1}\mathbf{m}_2 + \mathbf{m}_1, \mathrm{a}_1 + \mathrm{a}_2), \end{aligned} \tag{40}$$

$$U^{-1}(\mathbf{m}, a) \;=\; \begin{bmatrix} \mathbb{R}^{\mathrm{a}} & \mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbb{R}^{-\mathrm{a}} & -\mathbb{R}^{-\mathrm{a}}\mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix} \;=\; U(-\mathbb{R}^{-\mathrm{a}}\mathbf{m}, -\mathrm{a}). \tag{41}$$

Let us now summarize certain properties of group $\Gamma$:

- Both the translation group $\mathcal{T}$ and the rotation group $\mathcal{R}$ are subgroups of $\Gamma$ and their generic elements (pure translation or pure rotation) can respectively be written as

$$\text{pure translation} \;=\; U_{\mathcal{T}}(\mathbf{m}) \;=\; \begin{bmatrix} \boldsymbol{I} & \mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{42}$$

$$\text{pure rotation} \;=\; U_{\mathcal{R}}(a) \;=\; \begin{bmatrix} \mathbb{R}^{\mathrm{a}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{43}$$

- The generic element of $\Gamma$ given by (39) can always be written as

$$U(\mathbf{m}, a) \;=\; \begin{bmatrix} \mathbb{R}^{\mathrm{a}} & \mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix} \;=\; \begin{bmatrix} \boldsymbol{I} & \mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbb{R}^{\mathrm{a}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \;=\; U_{\mathcal{T}}(\mathbf{m})U_{\mathcal{R}}(a), \tag{44}$$

that is, as the product of a pure translation by a pure rotation. Moreover, decomposition (44) is unique. The utility of decomposition (44) is twofold: First, we have the possibility of writing every element of $\Gamma$ in a "normalized" form. Moreover, (44) simplifies the study of the action of $\Gamma$. For example, when necessary to verify if a given vector space $\mathcal{V}$ is invariant with respect to the action of some representation of $\Gamma$, it is sufficient to check the invariance of $\mathcal{V}$ with respect to pure rotations and pure translations.

- The combination of any element of $\Gamma$ and its inverse with a pure translation is still a pure translation:

$$U^{-1}(\mathbf{m}_1, a)U_{\mathcal{T}}(\mathbf{m}_2)U(\mathbf{m}_1, a) = \begin{bmatrix} \mathbb{R}^a & \mathbf{m}_1 \\ \mathbf{0} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{I} & \mathbf{m}_2 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbb{R}^a & \mathbf{m}_1 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \mathbb{R}^{-a}\mathbf{m}_2 \\ \mathbf{0} & 1 \end{bmatrix} = U_{\mathcal{T}}(\mathbb{R}^{-a}\mathbf{m}_2). \tag{45}$$

  In group theory $\mathcal{T}$ is called a *normal subgroup* of $\Gamma$. In (45) the resulting pure translation can be interpreted as the original translation rotated by $\mathbb{R}^{-a}$. Property (45) is used when searching for the irreducible representations of $\Gamma$. Note that a similar property *does not* hold for pure rotations.

Let us summarize what we have achieved so far:

- We found modulation and rotation groups, $\mathcal{T} = \{\mathbf{m} \mid \mathbf{m} \in \Lambda(\mathbf{Q})/\mathbb{Z}^2\}$ and $\mathcal{R} = \{\mathbf{I}, \mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^5\}$, respectively, with $\mathbf{Q} = \mathrm{diag}(1/16, 1/16)$ and $\mathbb{R}$ from (35). Remember that these operate in frequency domain.

- We put these groups together in order to get the final group $\Gamma = \{U(\mathbf{m}, a) \mid U(\mathbf{m}, a) = U_{\mathcal{T}}(\mathbf{m})U_{\mathcal{R}}(a), U_{\mathcal{T}}(\mathbf{m})$ from (42) and $U_{\mathcal{R}}(a)$ from (43)$\}$. Note that each element of group $\Gamma$ operates on a single point in frequency $\boldsymbol{\omega} = [\omega_1, \omega_2]^T$.

## 4.4 Determining the Irreducible Representations of $\Gamma$

To apply the theory presented so far, we need to find the irreducible representations of $\Gamma$. Since $\Gamma$ is a non-commutative group, finding its irreducible representations is not straightforward (see Appendix A in the first part of this work for details of representation theory). The construction used to obtain $\Gamma$ from $\mathcal{T}$ and $\mathcal{R}$ is well-known in group theory and $\Gamma$ is called the *semi-direct* product of $\mathcal{T}$ and $\mathcal{R}$. It is known that one can find every irreducible representation of $\Gamma$ by starting from the representations of $\mathcal{T}$ [6]. However, since using such results would require too much group machinery, one can follow a more intuitive approach that resembles the general one. Unfortunately, even this process is quite involved and we just give the irreducible representations.

It is possible to show that every irreducible representation of $\Gamma$ can be indexed by a vector $\mathbf{k} \in \mathbb{Z}^2/\Lambda\left(\mathbf{Q}^{-1}\right)$ and an integer $l = 0, \ldots, |\mathcal{R}|/N - 1$, with $N$ the first integer such that[3]

$$\mathbf{R}^N \mathbf{k} \;=\; \mathbf{k}. \tag{46}$$

Depending on $\mathbf{k}$, $N$ could be 1 in which case $l = 0, \ldots, 5$ (for $\mathbf{k} = [0,0]^T$), $N = 3$ with $l = 0, 1$ (for example for $\mathbf{k} = [8,0]^T$) and $N = 6$ with $l = 0$ (for example for $\mathbf{k} = [1,0]^T$).

We also need the following representation of $\mathcal{T}$, indexed by $\mathbf{k} \in \mathbb{Z}^2/\Lambda\left(\mathbf{Q}^{-1}\right)$

$$\pi_{\mathbf{k}}(U_{\mathcal{T}}(\mathbf{m})) \;\triangleq\; \begin{bmatrix} \exp(-j2\pi\mathbf{k}^T\mathbf{m}) & & & \\ & \exp(-j2\pi\mathbf{k}^T \mathbb{R}\mathbf{m}) & & \\ & & \ddots & \\ & & & \exp(-j2\pi\mathbf{k}^T \mathbb{R}^{\mathrm{N}-1}\mathbf{m}) \end{bmatrix}. \tag{47}$$

The above matrix could be a scalar, a $3 \times 3$ or a $6 \times 6$ matrix, depending on $\mathbf{k}$.

Now we give the form of the generic irreducible representation of $\Gamma$ relative to $\mathbf{k}$ and $l$, that is,

$$\begin{aligned} \pi_{\mathbf{k},l}(U_{\mathcal{T}}(\mathbf{m})) \;&=\; \pi_{\mathbf{k}}(U_{\mathcal{T}}(\mathbf{m})), \\ \pi_{\mathbf{k},l}(U_{\mathcal{R}}(a)) \;&=\; \exp(-j2\pi la/6)\, \mathbf{T}_N^a, \end{aligned} \tag{48}$$

with $\mathbf{k} \in \mathbb{Z}^2/\Lambda\left(\mathbf{Q}^{-1}\right)$ and $l = 0, \ldots, |\mathcal{R}|/N - 1$, and $\mathbf{T}_N$ being the circular translation $N \times N$ matrix

$$\mathbf{T}_N \;\triangleq\; \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{49}$$

Note that, since every element of $\Gamma$ can be written as a product of an element of $\mathcal{T}$ with one element of $\mathcal{R}$, from (48) and the fact that $\pi_{\mathbf{k},l}$ is a homomorphism (see Appendix A in the first part of this work), one can obtain the value of the irreducible representation $\pi_{\mathbf{k},l}$ for the generic element of $\Gamma$ as

$$\begin{aligned} &\pi_{\mathbf{k},l}(\mathbb{R}^a \cdot + \mathbf{m}) \;=\; \pi_{\mathbf{k},l}(U(\mathbf{m},a)) \;=\; \pi_{\mathbf{k},l}(U_{\mathcal{T}}(\mathbf{m})U_{\mathcal{R}}(a)) \;= \\ &\pi_{\mathbf{k},l}(U_{\mathcal{T}}(\mathbf{m}))\pi_{\mathbf{k},l}(U_{\mathcal{R}}(a)) \;=\; \pi_{\mathbf{k}}(\mathbf{m}) \exp(-j2\pi la/6)\, \mathbf{T}_N^a. \end{aligned} \tag{50}$$

## 4.5 Determining the Group of Modulations and Rotations Acting on a Vector

To complete the design, we need to put modulations and translations in a matrix form[4]. Note that now we find the structure of the new group $\Gamma^*$ in time domain[5]. To that end, we need to fix the filter support, in order to

[3] Note that (46) is an equality between classes of $\mathbb{Z}^2/\Lambda\left(\mathbf{Q}^{-1}\right)$ and it holds as soon as $\mathbf{R}^N\tilde{\mathbf{k}} = \tilde{\mathbf{k}} \pmod{\Lambda\left(\mathbf{Q}^{-1}\right)}$, with $\tilde{\mathbf{k}}$ any representative of class $\mathbf{k}$. It is possible to show that $N$ always divides $|\mathcal{R}| = 6$.

[4] In other words, we construct a matrix that describes the action of a modulation and a rotation on the whole signal/filter, as opposed to a single point we considered until now.

[5] Note that here we use $\Gamma^*, \mathcal{T}^*, \mathcal{R}^*, \mathcal{T}^*, \mathcal{R}^*$, to distinguish the group and its elements operating on the whole vector as opposed to a single point.

work with a finite-dimensional vector space. Of course, such a support has to be invariant with respect to the action of $\mathcal{R}$. An easy way to find a suitable support is to choose a "mother" set $A$ and define

$$B \;=\; A \,\cup\, \mathbf{R}A \,\cup \cdots \cup\, \mathbf{R}^5 A. \tag{51}$$

Then, we order the points of $B$ in order to map a function $f(\mathbf{n})$ defined on $B$ in a column vector $\mathbf{f}$. A particularly useful ordering can be obtained as follows: Consider the set of orbits resulting from the action of $\mathcal{R}$ on $B$[6]. The only orbit with just one element is $\mathcal{O}_0 \triangleq \{[0,0]^T\}$ while every other orbit has six elements. Choose any ordering for the other orbits and construct the vector $\mathbf{f}$ as

$$\mathbf{f} \triangleq \left[\begin{array}{c} f(\mathcal{O}_0) \\ \hline f(\mathcal{O}_1) \\ \hline f(\mathcal{O}_2) \\ \hline \vdots \end{array}\right], \tag{52}$$

where $f(\mathcal{O}_i)$ denotes the block vector with the values assumed by $f$ on the $i$th orbit. To order the points inside a given orbit $\mathcal{O}_i$, choose as the first point any $\mathbf{o}_r \in \mathcal{O}_i$ and order the other points as $\mathbf{o}_r, \mathbf{R}^{-1}\mathbf{o}_r, \ldots, \mathbf{R}^{-5}\mathbf{o}_r$.

Let us show an example of construction of vector $\mathbf{f}$. As the first orbit, choose the only orbit with one element $\mathcal{O}_0 = \{[0,0]^T\}$. Therefore, the first element of $\mathbf{f}$ is $f(0,0)$. To get the second orbit, choose another point (for example $[0,1]^T$) and take all the vectors of the form $\mathbf{R}^{-k}[0,1]^T$, $k = 0, \ldots, 5$, that is

$$\mathcal{O}_1 \;=\; \left\{[0,1]^T, [-1,0]^T, [-1,-1]^T, [0,-1]^T, [1,0]^T, [1,1]^T\right\}. \tag{53}$$

To find the third orbit choose, for example, $[0,2]$ to obtain $\mathbf{f}$ as

$$\mathbf{f} \;=\; \left[\begin{array}{c} f(0,0) \\ \hline f(0,1) \\ f(-1,0) \\ f(-1,-1) \\ f(0,-1) \\ f(1,0) \\ f(1,1) \\ \hline f(0,2) \\ f(-2,0) \\ f(-2,-2) \\ \vdots \end{array}\right]. \tag{54}$$

The procedure is iterated until all the points of $B$ are used.

We now need to find the new group elements consisting of matrices performing modulations and rotations of the whole vector $\mathbf{f}$. With ordering as above, these matrices assume a particularly simple form.

[6] An orbit of a point $\mathbf{n}$ is obtained by applying the elements of $\mathcal{R}$ to $\mathbf{n}$.

- Modulations correspond to a point-by-point multiplication of the function $\mathbf{f}$ by complex exponentials. The resulting matrix is diagonal.

- The effect of rotation $\mathbf{R}$ is to circularly rotate points of every orbit. The corresponding matrix is block diagonal, with every block a circular translation matrix corresponding to one orbit.

As an example, let us construct the matrices related to rotations and modulations in the specific case of $\mathbf{f}$ from (54). By applying a rotation $\mathbf{R}$ to filter $f(\mathbf{n})$ one obtains a new signal $g(\mathbf{n})$ defined as

$$g(\mathbf{n}) = f(\mathbf{R}\mathbf{n}). \tag{55}$$

Let $\boldsymbol{g}$ be the vector corresponding to filter $g(\mathbf{n})$. It is possible to write the components of $\boldsymbol{g}$ as a function of $\mathbf{f}$ as follows:

$$\boldsymbol{g} = \left[\begin{array}{c} g(0,0) \\ \hline g(0,1) \\ g(-1,0) \\ g(-1,-1) \\ g(0,-1) \\ g(1,0) \\ g(1,1) \\ \hline g(0,2) \\ \vdots \end{array}\right] = \left[\begin{array}{c} f(0,0) \\ \hline f(1,1) \\ f(0,1) \\ f(-1,0) \\ f(-1,-1) \\ f(0,-1) \\ f(1,0) \\ \hline f(2,2) \\ \vdots \end{array}\right] = U^*_{\mathcal{R}}(a) \left[\begin{array}{c} f(0,0) \\ \hline f(0,1) \\ f(-1,0) \\ f(-1,-1) \\ f(0,-1) \\ f(1,0) \\ f(1,1) \\ \hline f(0,2) \\ \vdots \end{array}\right]. \tag{56}$$

Then, matrix $U^*_{\mathcal{R}}(a)$ is

$$U^*_{\mathcal{R}}(a) = \begin{bmatrix} 1 & & & \\ & \mathbf{T}_6^a & & \\ & & \ddots & \\ & & & \mathbf{T}_6^a \end{bmatrix}_{169\times 169}. \tag{57}$$

In (57), empty entries are understood to be zero. We now construct the matrix relative to a modulation by $\exp(-j2\pi\mathbf{m}^T\mathbf{n})$. By modulating $f(\mathbf{n})$ one obtains a new signal $g(\mathbf{n})$ defined as

$$g(\mathbf{n}) = f(\mathbf{n})\exp(-j2\pi\mathbf{m}^T\mathbf{n}). \tag{58}$$

The corresponding vector $\boldsymbol{g}$ is

$$\boldsymbol{g} = \left[\begin{array}{c} g(0,0) \\ \hline g(0,1) \\ g(-1,0) \\ g(-1,-1) \\ g(0,-1) \\ g(1,0) \\ g(1,1) \\ \hline g(0,2) \\ \vdots \end{array}\right] = \left[\begin{array}{ll} \exp(-j2\pi\mathbf{m}^T[0,0]^T) & f(0,0) \\ \hline \exp(-j2\pi\mathbf{m}^T[1,1]^T) & f(1,1) \\ \exp(-j2\pi\mathbf{m}^T[0,1]^T) & f(0,1) \\ \exp(-j2\pi\mathbf{m}^T[-1,0]^T) & f(-1,0) \\ \exp(-j2\pi\mathbf{m}^T[-1,-1]^T) & f(-1,-1) \\ \exp(-j2\pi\mathbf{m}^T[0,-1]^T) & f(0,-1) \\ \exp(-j2\pi\mathbf{m}^T[1,0]^T) & f(1,0) \\ \hline \exp(-j2\pi\mathbf{m}^T[2,2]^T) & f(2,2) \\ \vdots & \end{array}\right] = U^*_{\mathcal{T}}(\mathbf{m})\mathbf{f}. \tag{59}$$

Then, the matrix $U^*_{\mathcal{T}}(\mathbf{m})$ is

$$U^*_{\mathcal{T}}(\mathbf{m}) = \begin{bmatrix} 1 & & & & \\ & \exp(-j2\pi\mathbf{m}^T[1,1]^T) & & & \\ & & \exp(-j2\pi\mathbf{m}^T[0,1]^T) & & \\ & & & \exp(-j2\pi\mathbf{m}^T[-1,0]^T) & \\ & & & & \ddots \end{bmatrix}_{169\times 169} . \tag{60}$$

Therefore, every matrix of $\Gamma^*$, being the product of $U^*_{\mathcal{R}}(a)$ and $U^*_{\mathcal{T}}(\mathbf{m})$ is in a block-diagonal form, that is, $\Gamma^*$ is of the form

$$\Gamma^* = \{U^*(\mathbf{m},a) \mid U^*(\mathbf{m},a) = U^*_{\mathcal{T}}(\mathbf{m})U^*_{\mathcal{R}}(a), U^*_{\mathcal{T}}(\mathbf{m}) \text{ from (60) and } U^*_{\mathcal{R}}(a) \text{ from (57)}\}. \tag{61}$$

Note that these are operating on the whole vector $\mathbf{f}$, not a single point only as in (44). Note also that we have constructed matrices $U^*(\mathbf{m},a) = U^*_{\mathcal{T}}(\mathbf{m})U^*_{\mathcal{R}}(a)$ which are block-diagonal and is thus well suited for finding representations. We do that first for translations and then for rotations.

Let us first consider $U^*_{\mathcal{T}}(\mathbf{m})$. By comparing the block $M_{\mathbf{o}_r}(\mathbf{m})$ in (60) corresponding to a certain orbit $\mathcal{O}_r$ (excluding $\mathcal{O}_0$) with $\pi_{\mathbf{o}_r}(U_{\mathcal{T}}(\mathbf{m}))$ given in (47) we can see that

$$M_{\mathbf{o}_r}(\mathbf{m}) = \mathrm{diag}(\{\exp(-j2\pi\mathbf{m}^T\mathbf{R}^{-i}\mathbf{o}_r), i=0,\ldots,5\}) = \mathrm{diag}(\{\exp(-j2\pi\mathbf{o}_r{}^T\mathbb{R}^{\mathrm{i}}\mathbf{m}), \mathrm{i}=0,\ldots,5\}) = \pi_{\mathbf{o_r}}(\mathbf{m}), \tag{62}$$

since $\mathbf{m}^T\mathbf{R}^{-i}\mathbf{o}_r$ is a scalar. Moreover, by using (62) and the fact that $\mathbb{R}\mathbf{Q} = \mathbf{QN}$ (see the beginning of this section) it is possible to prove that if $\mathbf{o}'_r = \mathbf{o}_r + \mathbf{Q}^{-T}\mathbf{s}$, $\mathbf{s} \in \mathbb{Z}^2$, then $M_{\mathbf{o}_r} = M_{\mathbf{o}'_r}$. In other words, representation $M_{\mathbf{o}_r}$ depends only on the equivalence class of $\mathbb{Z}^2/\Lambda\left(\mathbf{Q}^{-T}\right)$ to which $\mathbf{o}_r$ belongs. This suggests to collect equivalent orbits by letting them appear next to each other in the ordering. In the following we assume such an ordering.

As a result, $U^*_{\mathcal{T}}(\mathbf{m})$ can be further expressed as

$$\begin{aligned} U^*_{\mathcal{T}}(\mathbf{m}) &= \begin{bmatrix} 1 & & & & & \\ & \pi_{\mathbf{o}_{r_1},0}(U_{\mathcal{T}}(\mathbf{m})) & & & & \\ & & \ddots & & & \\ & & & \pi_{\mathbf{o}_{r_1},0}(U_{\mathcal{T}}(\mathbf{m})) & & \\ & & & & \pi_{\mathbf{o}_{r_2},0}(U_{\mathcal{T}}(\mathbf{m})) & \\ & & & & & \ddots \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & \\ & \boldsymbol{I}_{r_1}\otimes\pi_{\mathbf{o}_{r_1},0}(U_{\mathcal{T}}\mathbf{m})) & & \\ & & \boldsymbol{I}_{r_2}\otimes\pi_{\mathbf{o}_{r_2},0}(U_{\mathcal{T}}\mathbf{m})) & \\ & & & \ddots \end{bmatrix}, \end{aligned} \tag{63}$$

where the dimension of $\boldsymbol{I}_{r_i}$ for each distinct orbit tells us how many times a particular block appears. What the above expression means is that the matrix $U^*_{\mathcal{T}}$ is in a way its own representation.

For rotations, we can follow the same path, namely

$$
\begin{aligned}
U^*_{\mathcal{R}}(a) &= \begin{bmatrix} 1 & & & & & \\ & \pi_{\mathbf{o}_{r_1},l}(U_{\mathcal{R}}(a)) & & & & \\ & & \ddots & & & \\ & & & \pi_{\mathbf{o}_{r_1},l}(U_{\mathcal{R}}(a)) & & \\ & & & & \pi_{\mathbf{o}_{r_2},l}(U_{\mathcal{R}}(a)) & \\ & & & & & \ddots \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ & \boldsymbol{I}_{r_1} \otimes \pi_{\mathbf{o}_{r_1},l}(U_{\mathcal{R}}(a)) & & \\ & & \boldsymbol{I}_{r_2} \otimes \pi_{\mathbf{o}_{r_2},l}(U_{\mathcal{R}}(a)) & \\ & & & \ddots \end{bmatrix}.
\end{aligned} \tag{64}
$$

Therefore, the final element of group $\Gamma^*$ as well as its representation can be written as

$$
\begin{bmatrix} 1 & & & \\ & \boldsymbol{I}_{r_1} \otimes \pi_{\mathbf{o}_{r_1},0}(U_{\mathcal{T}}(\mathbf{m})) & & \\ & & \boldsymbol{I}_{r_2} \otimes \pi_{\mathbf{o}_{r_2},0}(U_{\mathcal{T}}(\mathbf{m})) & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \boldsymbol{I}_{r_1} \otimes \pi_{\mathbf{o}_{r_1},l}(U_{\mathcal{R}}(a)) & & \\ & & \boldsymbol{I}_{r_2} \otimes \pi_{\mathbf{o}_{r_2},l}(U_{\mathcal{R}}(a)) & \\ & & & \ddots \end{bmatrix} =
$$

$$
= \begin{bmatrix} 1 & & & \\ & \boldsymbol{I}_{r_1} \otimes \pi_{\mathbf{o}_{r_1},0}(U_{\mathcal{T}}(\mathbf{m}))\pi_{\mathbf{o}_{r_1},l}(U_{\mathcal{R}}(a)) & & \\ & & \boldsymbol{I}_{r_2} \otimes \pi_{\mathbf{o}_{r_2},0}(U_{\mathcal{T}}(\mathbf{m}))\pi_{\mathbf{o}_{r_2},l}(U_{\mathcal{R}}(a)) & \\ & & & \ddots \end{bmatrix}.
$$

With the chosen ordering, the form of matrices of $\Gamma^*$ is so simple that we can compute their Fourier transform $\mathcal{U}^{ij}_{\mathbf{k},l}$ in closed form. Observe that, given the block-diagonal nature of $\Gamma^*$, we just need to compute the Fourier transform ${}^r\mathcal{U}^{ij}_{\mathbf{k},l}$ of the block relative to $\mathbf{o}_r$. Consider first $|\mathcal{O}_r| = 6$. Then $N = 6$ and $l = 0$. From (4) and orthogonality relations one obtains

$$
{}^r\mathcal{U}^{ij}_{\mathbf{k},0} = \frac{1}{256}\delta(\mathbf{o}_r - \mathbf{k})e_{ij} \tag{65}
$$

where $e_{ij}$ is a $6 \times 6$ matrix having 1 in row $i$, column $j$ and zero otherwise.

On the other hand, if $|\mathcal{O}_r| = 1$, that is, $\mathbf{o}_r = \mathbf{o}_1 = [0, 0]^T$, then $N = 1$ and $l = 0, \ldots, 5$, yielding

$$
{}^1\mathcal{U}^{ij}_{\mathbf{k},l} = \frac{1}{256 \cdot 6}\delta(\mathbf{k})\delta(l). \tag{66}
$$

By putting together expressions (65) and (66) one obtains the general form of $\mathcal{U}^{ij}_{\mathbf{k},l}$

$$
\mathcal{U}^{ij}_{\mathbf{k},l} = \begin{bmatrix} \ddots & & & \\ & 0 & & \\ & & \boldsymbol{I} \otimes {}^r\mathcal{U}^{ij}_{\mathbf{k},l} & \\ & & & 0 \\ & & & & \ddots \end{bmatrix}, \tag{67}
$$

that is, $\mathcal{U}^{ij}_{\mathbf{k},l}$ is a block diagonal matrix, zero everywhere but on the blocks relative to $\mathbf{o}_r = \mathbf{k} \pmod{\Lambda\left(\mathbf{Q}^{-T}\right)}$. In (67) $\boldsymbol{I}$ is an identity matrix of suitable dimensions. Note, from (67), that the dimension of $\mathcal{V}^1_{\mathbf{k},l}$ is equal to the number of orbits relative to the same representation $\pi_{\mathbf{k},l}$. Remember that $\dim(\mathcal{V}^1_{\mathbf{k},l}) \geq \dim(\pi_{\mathbf{k},l})$. This implies that the number of orbits relative to representation $\pi_{\mathbf{k},l}$ must be greater or equal than $\dim(\pi_{\mathbf{k},l})$, that is, the support must be big enough.

### 4.6 Finding the Filter

In this example we used the support of Figure 7(a) that has been determined via (51) with $A$ suitably chosen in order to have enough orbits. Then, we applied Algorithm 1 in order to obtain the filter of Figure 7(b). As a starting filter we used

$$h(x,y) \triangleq \exp(-0.0625(x^2 + 0.01y^2))\sin(2\pi x/14). \tag{68}$$

The purpose of the gaussian part is to give a lowpass shape to the filter, while the modulation with the sinusoid has been used to make the orthogonalization of $h(x,y)$ easier. Indeed, without the sinusoid, $h(x,y)$ would be nonnegative everywhere and such a function cannot be, even approximately, orthogonal to its own rotation. Because of this, although the algorithm finds the orthogonal filter that is closest to $h(x,y)$, filter (68) could be too far from the set of the filters orthogonal with respect to the action of $\Gamma^*$ and the output of Algorithm 1 would not be meaningful.

## A Proofs

**Proof of correctness of (8):** Using (8) in the left-hand side of (6), one obtains

$$\begin{aligned}\{\hat{\boldsymbol{B}}, \mathcal{U}^{ij}_\omega \hat{\boldsymbol{B}}\} &= \sum_{i_1,\omega_1}\sum_{i_2,\omega_2}\{\mathcal{U}^{i_1 1}_{\omega_1}\bar{B}_{\omega_1,i_1}, \mathcal{U}^{ij}_\omega \mathcal{U}^{i_2 1}_{\omega_2}\bar{B}_{\omega_2,i_2}\}, \\ &= \sum_{i_1,\omega_1}\sum_{i_2,\omega_2}\{\bar{B}_{\omega_1,i_1}, \mathcal{U}^{1 i_1}_{\omega_1}\mathcal{U}^{ij}_\omega \mathcal{U}^{i_2 1}_{\omega_2}\bar{B}_{\omega_2,i_2}\}.\end{aligned} \tag{69}$$

Because of (7), in (69) the only terms that remain are for $i_1 = i$, $i_2 = j$ and $\omega_1 = \omega_2 = \omega$, and (69) can be rewritten as

$$\{\boldsymbol{B}, \mathcal{U}^{ij}_\omega \boldsymbol{B}\} = \{\bar{B}_{\omega,i}, \mathcal{U}^{1i}_\omega \mathcal{U}^{ij}_\omega \mathcal{U}^{j1}_\omega \bar{B}_{\omega,j}\}. \tag{70}$$

Using (7) once more, we get

$$\{\boldsymbol{B}, \mathcal{U}^{ij}_\omega \boldsymbol{B}\} = \{\bar{B}_{\omega,i}, \mathcal{U}^{11}_\omega \bar{B}_{\omega,j}\} = \{\bar{B}_{\omega,i}, \bar{B}_{\omega,j}\} = \boldsymbol{I}\delta_{i-j}. \tag{71}$$

In (71) we used the fact that $\mathcal{U}^{11}_\omega \bar{B}_{\omega,j} = \bar{B}_{\omega,j}$ since $\bar{B}_{\omega,j}$ belongs to $\mathcal{V}^1_\omega$ and $\mathcal{U}^{11}_\omega$ is its projection operator. ■

The above holds since $\mathcal{U}_\omega^{11}$ projects $\bar{B}_{\omega,j}$ onto the same space $\mathcal{V}_\omega^j$ and $\bar{B}_{\omega,i}$ are all orthogonal to one another.

**Proof of Property 2:** We need to prove two facts: that $\mathcal{V}_\omega^i = \mathcal{U}_\omega^{ij}\mathcal{V}_\omega^j$ and that $\ker\left(\mathcal{U}_\omega^{ij}\right) \cap \mathcal{V}_\omega^j = \{0\}$.

In order to prove the first fact we show that if $\boldsymbol{v} \in \mathcal{V}_\omega^j$ then $\mathcal{U}_\omega^{ij}\boldsymbol{v} \in \mathcal{V}_\omega^i$ and that if $\boldsymbol{v}_1 \in \mathcal{V}_\omega^i$ then there exists $\boldsymbol{v} \in \mathcal{V}_\omega^j$ such that $\boldsymbol{v}_1 = \mathcal{U}_\omega^{ij}\boldsymbol{v}$. Indeed, if $\boldsymbol{v} \in \mathcal{V}_\omega^j$, then

$$\mathcal{U}_\omega^{ii}\left(\mathcal{U}_\omega^{ij}\boldsymbol{v}\right) = \mathcal{U}_\omega^{ij}\boldsymbol{v} \tag{72}$$

because of Property 1. Equation (72) says that the projection of $\mathcal{U}_\omega^{ij}\boldsymbol{v}$ on $\mathcal{V}_\omega^i$ is equal to $\mathcal{U}_\omega^{ij}\boldsymbol{v}$ itself, that is, $\mathcal{U}_\omega^{ij}\boldsymbol{v}$ belongs to $\mathcal{V}_\omega^i$. To prove that for each $\boldsymbol{v}_1 \in \mathcal{V}_\omega^i$ there exists a $\boldsymbol{v} \in \mathcal{V}_\omega^j$ such that $\boldsymbol{v}_1 = \mathcal{U}_\omega^{ij}\boldsymbol{v}$, it is sufficient to verify that $\boldsymbol{v} \triangleq \mathcal{U}_\omega^{ji}\boldsymbol{v}_1$ works. Indeed

$$\mathcal{U}_\omega^{ij}\mathcal{U}_\omega^{ji}\boldsymbol{v}_1 = \mathcal{U}_\omega^{ii}\boldsymbol{v}_1 = \boldsymbol{v}_1 \tag{73}$$

and the first fact is proved.

To prove the second fact let us suppose that there exists a $\boldsymbol{v}$ from $\mathcal{V}_\omega^j$ which also belongs to the kernel of $\mathcal{U}_\omega^{ij}$, that is

$$\mathcal{U}_\omega^{ij}\boldsymbol{v} = 0. \tag{74}$$

By multiplying (74) by $\mathcal{U}_\omega^{ji}$ one obtains

$$\mathcal{U}_\omega^{ji}\mathcal{U}_\omega^{ij}\boldsymbol{v} = \mathcal{U}_\omega^{jj}\boldsymbol{v} = 0. \tag{75}$$

However, remember that $\boldsymbol{v} \in \mathcal{V}_\omega^j$. This immediately means that

$$\boldsymbol{v} = 0, \tag{76}$$

that is, the only vector in $\ker\left(\mathcal{U}_\omega^{ij}\right) \cap \mathcal{V}_\omega^j$ is the zero vector. ■

To prove Property 3 we need the following lemma:

**Lemma A.1** *Matrices $\mathcal{U}_\omega^{ij}$ are unitary transformations between $\mathcal{V}_\omega^j$ and $\mathcal{V}_\omega^i$.*

**Proof:** Let $\boldsymbol{b}, \boldsymbol{c} \in \mathcal{V}_\omega^j$, then

$$\langle \mathcal{U}_\omega^{ij}\boldsymbol{b}, \mathcal{U}_\omega^{ij}\boldsymbol{c}\rangle = \langle \boldsymbol{b}, \mathcal{U}_\omega^{ji}\mathcal{U}_\omega^{ij}\boldsymbol{c}\rangle = \langle \boldsymbol{b}, \mathcal{U}_\omega^{jj}\boldsymbol{c}\rangle = \langle \boldsymbol{b}, \boldsymbol{c}\rangle. \tag{77}$$

■

Note that (77) still holds if vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are replaced with vector sets.

**Proof of Property 3:** In Algorithm 2 vector set $\boldsymbol{B}$ is projected on spaces $\mathcal{V}_\omega^j$ to obtain vector sets $\boldsymbol{B}_{j\omega}$. Such vector sets are mapped in $\mathcal{V}_\omega^1$, via $\mathcal{U}_\omega^{1j}$, to obtain $\boldsymbol{B}_{j\omega}^1$. Note that the original vector set can be expressed as

$$\sum_{\omega,j} \mathcal{U}_\omega^{j1} \boldsymbol{B}_{j\omega}^1 \;=\; \sum_{\omega,j} \mathcal{U}_\omega^{j1} \mathcal{U}_\omega^{1j} \boldsymbol{B}_{j\omega} \;=\; \sum_{\omega,j} \mathcal{U}_\omega^{jj} \boldsymbol{B}_{j\omega} \;=\; \boldsymbol{B} \tag{78}$$

because $\mathcal{V}$ is direct sum of vector spaces $\mathcal{V}_\omega^j$.

In the second step of Algorithm 2, for each $\omega$, vector sets $\boldsymbol{B}_{j,\omega}^1$ are "clustered" together and orthogonalized. By looking at Algorithm 2 and Property 7 from [1] note that $\boldsymbol{B}_{j,\omega}^1$ are orthogonalized via the SVD for vector set. Because of Property 8 the new vector sets $\hat{\boldsymbol{B}}_{j,\omega}^1$ are such that the following quantity:

$$\sum_j ||\hat{\boldsymbol{B}}_{j,\omega}^1 - \boldsymbol{B}_{j,\omega}^1||^2 \tag{79}$$

is minimized for each $\omega$. The new vector set $\hat{\boldsymbol{B}}$ is

$$\hat{\boldsymbol{B}} \;=\; \sum_{\omega,j} \mathcal{U}_\omega^{j1} \hat{\boldsymbol{B}}_{j\omega}^1. \tag{80}$$

The distance between $\boldsymbol{B}$ and $\hat{\boldsymbol{B}}$ can be written as

$$||\sum_\omega \sum_j \mathcal{U}_\omega^{j1} \hat{\boldsymbol{B}}_{j\omega}^1 - \mathcal{U}_\omega^{j1} \boldsymbol{B}_{j\omega}^1||^2. \tag{81}$$

Because the terms inside the sum of (81) are orthogonal to one another (they belong to vector spaces $\mathcal{V}_\omega^j$, orthogonal to one another) one can apply the Pythagorean theorem to obtain

$$\sum_\omega \sum_j ||\mathcal{U}_\omega^{j1} \left( \hat{\boldsymbol{B}}_{j\omega}^1 - \boldsymbol{B}_{j\omega}^1 \right)||^2. \tag{82}$$

Since each $\mathcal{U}_\omega^{j1}$ is unitary (Lemma A.1) (82) can be rewritten as

$$\begin{aligned} \sum_\omega \sum_j ||\mathcal{U}_\omega^{j1} \left( \hat{\boldsymbol{B}}_{j\omega}^1 - \boldsymbol{B}_{j\omega}^1 \right)||^2 \;&=\; \sum_\omega \sum_j ||\left( \hat{\boldsymbol{B}}_{j\omega}^1 - \boldsymbol{B}_{j\omega}^1 \right)||^2 \\ &=\; \sum_\omega ||\sum_j \left( \hat{\boldsymbol{B}}_{j\omega}^1 - \boldsymbol{B}_{j\omega}^1 \right)||^2. \end{aligned} \tag{83}$$

In (83) it has been possible to bring $\sum_j$ inside the norm again because of the Pythagorean theorem. Since each term in $\sum_\omega$ is independently minimized, the global sum is minimized too. ■

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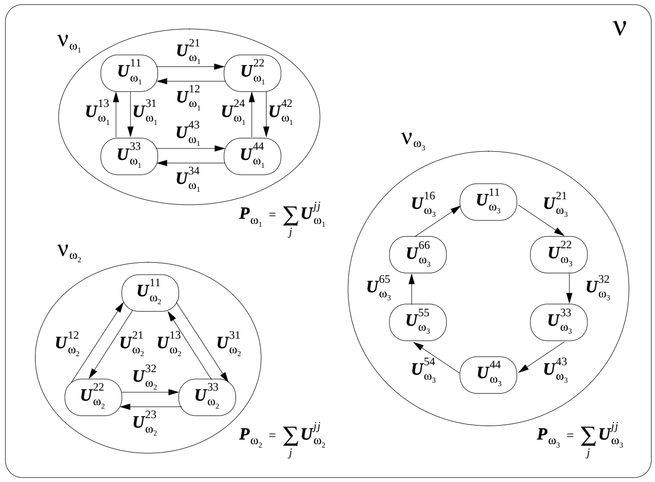



Figure 1: Subdivision of the vector space $\mathcal{V}$ according to matrices $\mathcal{U}_{\omega}^{ij}$ in (7). Note that not all the branches are shown in order not to clog the figure.

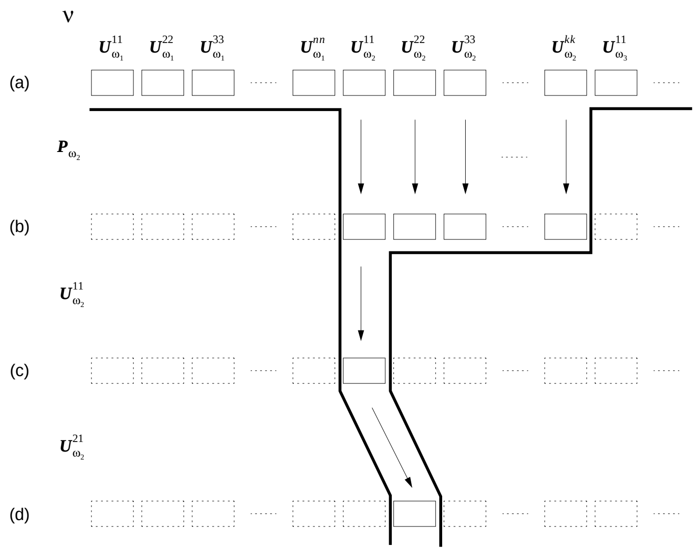



Figure 2: Structure of $\mathcal{V}$ when a suitable basis is chosen. (a) Block subdivision of a vector from $\mathcal{V}$. (b) The action of projection $\mathcal{P}_{\omega_2}$ on the vector in (a) corresponds to keeping only the blocks corresponding to $\omega_2$. (c) The action of $\mathcal{U}^{11}_{\omega_2}$ on vector in (b) corresponds to keeping only the first block. (d) The action of $\mathcal{U}^{21}_{\omega_2}$ corresponds to moving the first block into the second one.

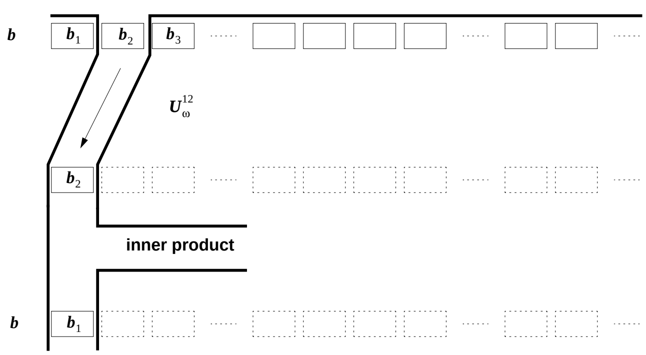



Figure 3: Graphical interpretation of the scalar product $\langle \boldsymbol{b}, \mathcal{U}^{ij}_{\omega} \boldsymbol{b} \rangle$ as a scalar product between blocks of $\boldsymbol{b}$.

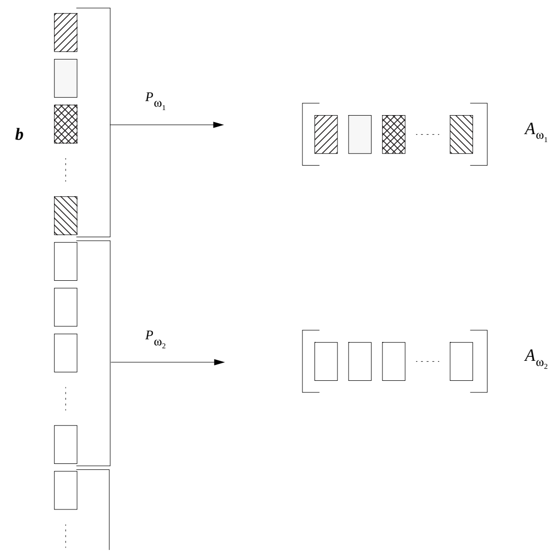



Figure 4: Parameterization of a vector $\boldsymbol{b}$ orthogonal with respect to the action of $\Gamma$. First vector $\boldsymbol{b}$ is decomposed in blocks, then the blocks are organized as columns of matrices. Since each matrix has to be orthogonal, it can be identified by its Givens' rotations.

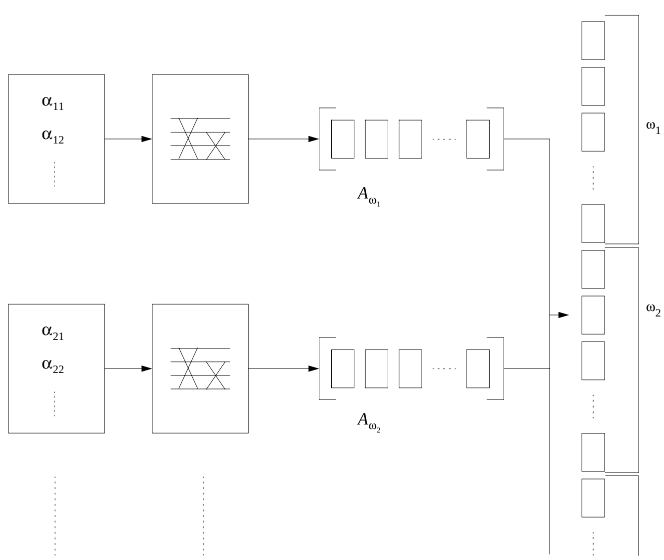



Figure 5: Construction of $\boldsymbol{b}$ from the Givens' rotations $\alpha_{kn}$. Each set of Givens' rotations gives rise to an orthogonal matrix whose columns are used as the blocks of $\boldsymbol{b}$.

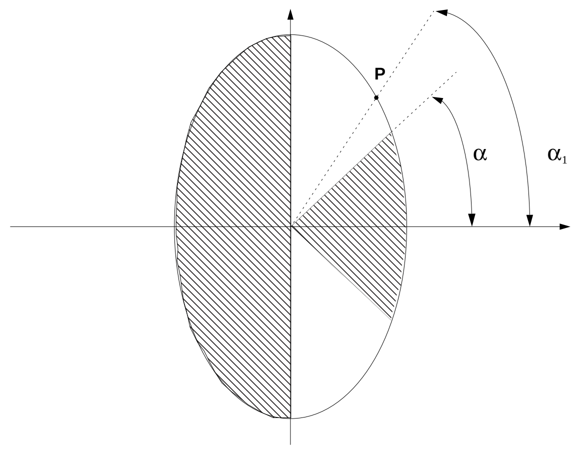
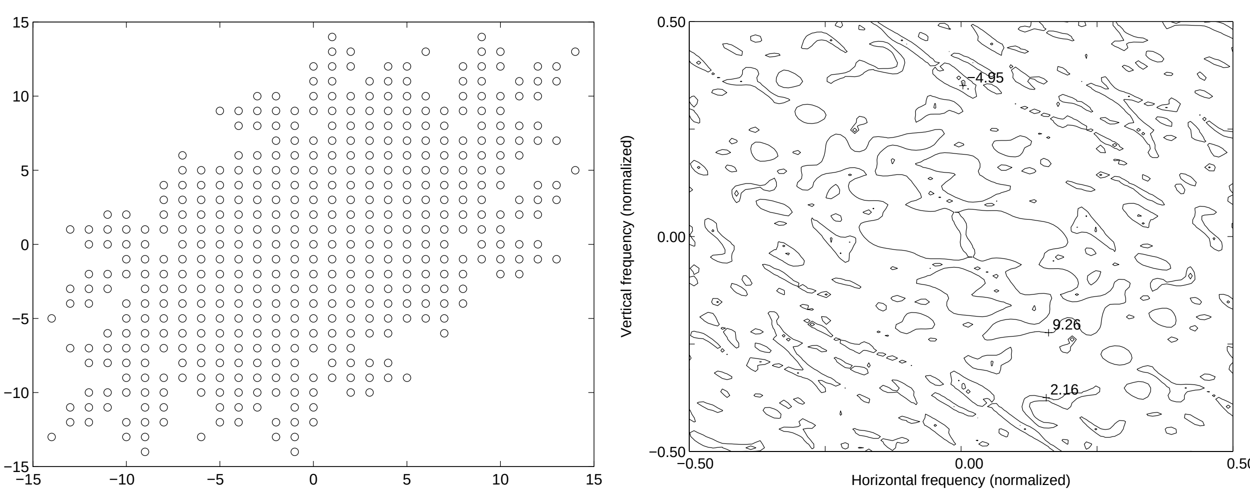



Figure 6: Geometric interpretation of (28).




(a) (b)

Figure 7: (a) Filter support used in the example in the text. (b) Frequency response of the filter designed with our technique. The contour levels plotted are at 20, 15 and 12 dB.