# Multidimensional Orthogonal Filter Bank Characterization and Design Using the Cayley Transform

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***Abstract*—We present a complete characterization and design of orthogonal infinite impulse response (IIR) and finite impulse response (FIR) filter banks in any dimension using the Cayley transform (CT). Traditional design methods for one-dimensional orthogonal filter banks cannot be extended to higher dimensions directly due to the lack of a multidimensional (MD) spectral factorization theorem. In the polyphase domain, orthogonal filter banks are equivalent to paraunitary matrices and lead to solving a set of *nonlinear* equations. The CT establishes a *one-to-one* mapping between paraunitary matrices and para-skew-Hermitian matrices. In contrast to the paraunitary condition, the para-skew-Hermitian condition amounts to *linear* constraints on the matrix entries which are much easier to solve. Based on this characterization, we propose efficient methods to design MD orthogonal filter banks and present new design results for both IIR and FIR cases.**

***Index Terms*—Cayley transform (CT), filter banks, multidimensional (MD) filter banks, nonseparable filter design, orthogonal filter banks, paraunitary, polyphase.**

## I. Introduction

OVER the last decade, the theory and applications of filter banks have grown rapidly [1]–[6]. Among them, orthogonal filter banks received particular attention due to their useful properties. First, orthogonality implies energy preservation, which guarantees that the energy of errors generated by transmission or quantization will not be amplified. Second, under certain conditions, orthogonal filter banks can be used to construct orthonormal wavelet bases [7].

There are two types of orthogonal filter banks: *infinite impulse response* (IIR) filter banks and *finite impulse response* (FIR) filter banks. For simplicity, we consider only filter banks with real coefficients. Orthogonal IIR filter banks have greater design freedom and, thus, generally offer better frequency selectivity. Herley and Vetterli considered the theory and design of one-dimensional (1-D) two-channel orthogonal IIR filter banks [8]. Selesnick proposed explicit formulas for two classes of 1-D two-channel orthogonal IIR filters [9]. However, their design methods need spectral factorization and, hence, cannot be extended to higher dimensions directly.

Orthogonal FIR filter banks are easier to implement and, hence, more popular. In the 1-D two-channel filter bank case, there exist several filter design methods. Among them, designs based on spectral factorizations [10] and designs based on lattice factorizations [11] are the most effective and widely used. The first method, which was proposed by Smith and Barnwell [10], designs the autocorrelation sequence of a filter and then obtains that filter via spectral factorization. This method was used by Daubechies to construct the celebrated family of orthogonal compactly supported wavelets [7]. However, as the size of the filter grows, spectral factorization becomes numerically ill conditioned. Moreover, this method is difficult to extend to higher dimensions due to the lack of a multidimensional (MD) factorization theorem.

The second method, which was proposed by Vaidyanathan and Hoang [11], formulates the filter design problem as that of a polyphase transform matrix which has to be a *paraunitary*[1] matrix $\mathbf{U}(z)$, such that

$$\mathbf{U}(z)\mathbf{U}^T(z^{-1}) = \mathbf{I}, \quad \text{for real coefficients} \tag{1}$$

where $\mathbf{I}$ is an identity matrix. These authors provided a complete characterization of paraunitary FIR matrices for 1-D filter banks via a lattice factorization. However, in multiple dimensions, the lattice structure is *not* a complete characterization.

In multiple dimensions, there are two types of filter banks: separable and nonseparable filter banks. Transfer functions of a separable filter bank are products of multiple 1-D filters. Therefore, tensor products can be used to construct separable filter banks from 1-D filter banks. In contrast to separable filter banks, nonseparable filter banks are designed based on the MD structure directly, resulting in more freedom and better frequency selectivity. In addition, nonseparable filter banks lead to flexible directional decomposition of MD data [12]. Therefore, nonseparable filter banks have received more interest in recent years.

Due to complexity, it is a challenging task to design nonseparable MD orthogonal filter banks. In the IIR case, to the best of our knowledge, there is no existing literature addressing the design problem. In the FIR case, to avoid spectral factorization, Kovačević and Vetterli used the lattice structure to pa-

[1]A paraunitary matrix is an extension of a unitary matrix when the matrix entries are Laurent polynomials. Paraunitary matrices are unitary on the unit circle.

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rameterize the paraunitary matrices in MD and successfully designed specific two-dimensional (2-D) and three-dimensional (3-D) nonseparable orthogonal FIR filter banks [13]. However, their method could not find all solutions since the MD lattice structure is not a complete characterization. Recently, Delgosha and Fekri derived a complete factorization for 2-D orthogonal FIR filter banks based on degree-one IIR building blocks [14]. Unlike our work, the focus of their work is to find a factorization of a *given* orthogonal FIR filter bank.

In this work, we propose a complete characterization of MD orthogonal filter banks and a novel design method for orthogonal IIR and FIR filter banks using the *Cayley transform* (CT) [15]. The CT of a matrix $\mathbf{U}(\boldsymbol{z})$ is defined as

$$\mathbf{H}(\boldsymbol{z}) = \big(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\big)^{-1}\big(\mathbf{I} - \mathbf{U}(\boldsymbol{z})\big). \tag{2}$$

The inverse of the CT is

$$\mathbf{U}(\boldsymbol{z}) = \big(\mathbf{I} + \mathbf{H}(\boldsymbol{z})\big)^{-1}\big(\mathbf{I} - \mathbf{H}(\boldsymbol{z})\big). \tag{3}$$

The CT is a matrix generalization of the bilinear transform [16, pp. 415–417], which is defined as $s = (1+z)^{-1}(1+z)$. The bilinear transform maps the imaginary axis of the complex $s$ plane onto the unit circle in the complex $z$ plane. It is widely used in signal processing theory, for example, to map continuous-time systems to discrete-time systems. The CT is a powerful tool to convert a nonlinear problem into a linear one and is widely used in control theory and Lie groups [17]. We will show that the CT maps a paraunitary matrix to a *para-skew-Hermitian*[2] (PSH) matrix $\mathbf{H}(\boldsymbol{z})$ that satisfies

$$\mathbf{H}(\boldsymbol{z}^{-1}) = -\mathbf{H}^{T}(\boldsymbol{z}), \quad \text{for real coefficients.} \tag{4}$$

Conversely, the inverse CT maps a PSH matrix to a paraunitary matrix. Therefore, the CT establishes a one-to-one mapping between a nonlinear Stiefel manifold of paraunitary matrices and the linear space of PSH matrices, as shown in Fig. 1.

Our key observation is that, in contrast to solving for the nonlinear paraunitary condition in (1), the PSH condition amounts to *linear* constraints on the matrix entries in (4), leading to an easier design problem. The basic idea is that we first design a PSH matrix and then map it back to a paraunitary matrix by the CT. This approach simplifies the design problem of orthogonal filter banks. However, there are three challenges in this design approach due to the matrix inversion term in the CT. The first challenge is how to guarantee that the matrix inverse exists. The second challenge is that the CT destroys the FIR property because of this term; that is, the CT of an FIR matrix is, in general, no longer FIR. Thus, for orthogonal FIR filter banks, we need to find a complete characterization of the PSH matrices such that their inverse CTs are FIR. The third challenge is how to impose certain filter bank conditions (such as vanishing moments) in the Cayley domain. In this paper, we address these issues, leading to a complete characterization and a novel design method for orthogonal IIR and FIR filter banks of any dimension and any number of channels.

The rest of the paper is organized as follows. In Section II, we study the link between orthogonal filter banks and the CT.

[2]A para-skew-Hermitian matrix is an extension of a skew-Hermitian matrix when the matrix entries are Laurent polynomials. Para-skew-Hermitian matrices are skew-Hermitian on the unit circle.

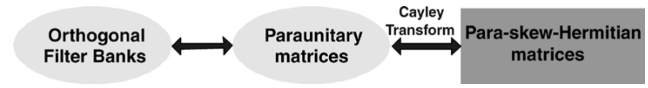



Fig. 1. One-to-one mapping between paraunitary matrices and para-skew-Hermitian matrices via the CT.

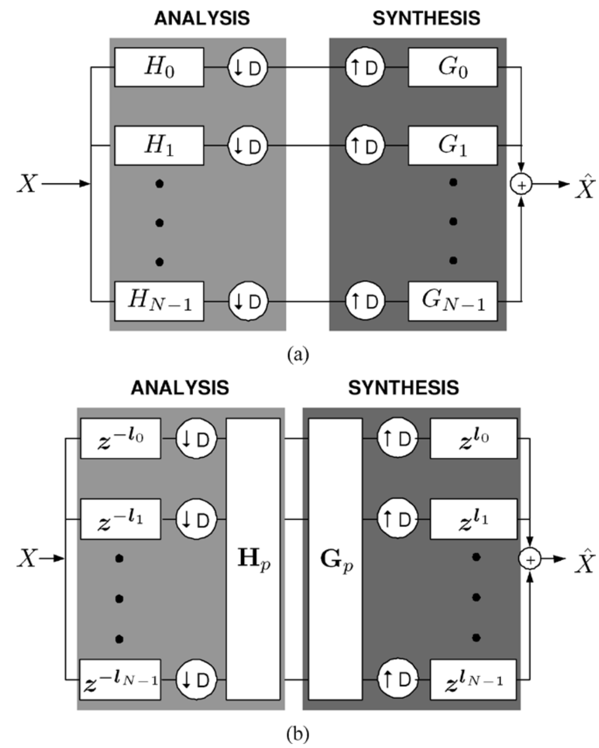



Fig. 2. (a) MD $N$-channel filter bank: $H_i$ and $G_i$ are MD analysis and synthesis filters, respectively. $\mathbf{D}$ is an $M \times M$ sampling matrix. (b) Polyphase representation: $\mathbf{H}_p$ and $\mathbf{G}_p$ are MD $N \times N$ analysis and synthesis polyphase transform matrices, respectively. $\{\boldsymbol{l}_j\}_{j=0}^{N-1}$ is the set of all integer vectors in $FPD(\mathbf{D})$.

The characterization and design of orthogonal IIR filter banks is given in Section III. The characterization and design of orthogonal FIR filter banks, including the general multiple-channel case and the particular two-channel case are given in Sections IV and V, respectively. We conclude in Section VI.

## II. Orthogonal Filter Banks and the CT

We start with notations. Throughout the paper, we will always refer to $M$ as the number of dimensions and $N$ as the number of channels. In MD, $\boldsymbol{z}$ stands for an $M$-dimensional variable $\boldsymbol{z} = [z_1, z_2, \ldots, z_M]^T$ and $\boldsymbol{z}^{-1}$ stands for $[z_1^{-1}, z_2^{-1}, \ldots, z_M^{-1}]^T$. Raising $\boldsymbol{z}$ to an $M$-dimensional integer vector $\boldsymbol{k} = [k_1, k_2, \ldots, k_M]^T$ yields $\boldsymbol{z}^{\boldsymbol{k}} = \prod_{i=1}^{M} z_i^{k_i}$. Raising $\boldsymbol{z}$ to an $M \times M$ integer matrix $\mathbf{D}$ yields $\boldsymbol{z}^{\mathbf{D}} = [\boldsymbol{z}^{\boldsymbol{d}_1}, \boldsymbol{z}^{\boldsymbol{d}_2}, \ldots, \boldsymbol{z}^{\boldsymbol{d}_M}]^T$, where $\boldsymbol{d}_i$ is the $i$th column of $\mathbf{D}$. In addition, $FPD(\mathbf{D})$ stands for the fundamental parallelepiped generated by $\mathbf{D}$, which is the set of all vectors $\mathbf{D}\boldsymbol{t}$ with $\boldsymbol{t} \in [0,1)^M$ [11].

Consider an MD $N$-channel filter bank, as shown in Fig. 2(a). We are interested in critically sampled filter banks in which the sampling rate is equal to the number of channels, i.e., $|\det \mathbf{D}| = N$. In the analysis and design of filter banks, polyphase representation is often used as it allows for time-invariant analysis in the polyphase domain as shown in Fig. 2(b). In the polyphase domain, the analysis and synthesis parts can be represented by $N \times N$ matrices $\mathbf{H}_p(z)$ and $\mathbf{G}_p(z)$, respectively. The analysis and synthesis filters are related to the corresponding polyphase matrices as [11]

$$H_i(z) = \sum_{\boldsymbol{l}_j \in \mathcal{N}(\mathbf{D})} z^{-\boldsymbol{l}_j} \{\mathbf{H}_p\}_{i,j}(z^{\mathbf{D}})$$
$$G_i(z) = \sum_{\boldsymbol{l}_j \in \mathcal{N}(\mathbf{D})} z^{\boldsymbol{l}_j} \{\mathbf{G}_p\}_{j,i}(z^{\mathbf{D}}), \text{ for } i=0,1,\ldots,N\text{-}1 \quad (5)$$

where $\mathcal{N}(\mathbf{D})$ is the set of all integer vectors in $FPD(\mathbf{D})$, and $\{\mathbf{A}\}_{i,j}$ is the $(i,j)$ entry of the matrix $\mathbf{A}$.

In the polyphase domain, the perfect reconstruction condition $\hat{X}(z) = X(z)$ is equivalent to $\mathbf{H}_p(z)\mathbf{G}_p(z) = \mathbf{I}$. Orthogonal condition additionally require $\mathbf{G}_p(z) = \mathbf{H}_p^T(z^{-1})$, and, thus, $\mathbf{H}_p(z)$ and $\mathbf{G}_p(z)$ are paraunitary matrices. Therefore, designing an orthogonal filter bank amounts to designing a paraunitary matrix. From now on, we denote $\mathbf{G}_p(z)$ by $\mathbf{U}(z)$ for convenience, and we assume that $\mathbf{U}(z)$ is an $N \times N$ paraunitary matrix, and $\mathbf{H}(z)$ is its CT.

As mentioned in Section I, the CT maps a paraunitary matrix to a PSH matrix. To use the CT, we must make sure that $\mathbf{I} + \mathbf{U}(z)$ is invertible as required in (2). If $\mathbf{I} + \mathbf{U}(z)$ is not invertible, we can adjust the filter bank by multiplying some filters with $-1$, to obtain an equivalent filter bank. Doing so is equivalent to multiplying the corresponding rows of $\mathbf{U}(z)$ with $-1$ to obtain an equivalent paraunitary matrix. In this way, we can generate $2^N$ equivalent $\mathbf{U}(z)$ from one $N$-channel orthogonal filter bank. The following proposition guarantees that among those $2^N$ equivalent matrices there exists at least one $\mathbf{U}(z)$ such that $\mathbf{I} + \mathbf{U}(z)$ is invertible.

*Proposition 1:* Suppose $\mathbf{U}(z)$ is an $N \times N$ matrix and $\Lambda$ is an $N \times N$ diagonal matrix whose diagonal entries are either 1 or $-1$. Then, there exists at least one $\Lambda$ such that $\mathbf{I} + \Lambda\mathbf{U}(z)$ is nonsingular.

*Proof:* See Appendix A. ■

By Proposition 1, we can always find an equivalent paraunitary polyphase matrix $\mathbf{U}(z)$ for any orthogonal filter bank such that $\mathbf{I} + \mathbf{U}(z)$ is invertible and, thus, its CT $\mathbf{H}(z)$ exists. To obtain $\mathbf{U}(z)$ from $\mathbf{H}(z)$ by the inverse CT, we must make sure that $\mathbf{I} + \mathbf{H}(z)$ is also invertible as required in (3). The following proposition shows $\mathbf{I} + \mathbf{H}(z) = 2(\mathbf{I} + \mathbf{U}(z))^{-1}$ and, hence, that the condition for $\mathbf{I} + \mathbf{H}(z)$ to be invertible is equivalent to the condition for $\mathbf{I} + \mathbf{U}(z)$ to be invertible.

*Proposition 2:* Suppose that $\mathbf{H}(z)$ is the CT of $\mathbf{U}(z)$. Then

$$\mathbf{H}(z) = 2(\mathbf{I} + \mathbf{U}(z))^{-1} - \mathbf{I}$$
$$\mathbf{U}(z) = 2(\mathbf{I} + \mathbf{H}(z))^{-1} - \mathbf{I}.$$

*Proof:* Using (2), we have

$$\begin{aligned}\mathbf{I} + \mathbf{H}(z) &= \mathbf{I} + (\mathbf{I} + \mathbf{U}(z))^{-1}(\mathbf{I} - \mathbf{U}(z)) \\ &= (\mathbf{I} + \mathbf{U}(z))^{-1}[(\mathbf{I} + \mathbf{U}(z)) + (\mathbf{I} - \mathbf{U}(z))] \\ &= 2(\mathbf{I} + \mathbf{U}(z))^{-1}.\end{aligned}$$

■

From Proposition 1 and Proposition 2, we can associate any orthogonal filter bank with a paraunitary matrix $\mathbf{U}(z)$ and a PSH matrix $\mathbf{H}(z)$ such that both $\mathbf{I} + \mathbf{U}(z)$ and $\mathbf{I} + \mathbf{H}(z)$ are invertible. Therefore, in the rest of the paper, without loss of generality, we can assume that the CT of the polyphase matrix $\mathbf{U}(z)$ and the inverse CT of $\mathbf{H}(z)$ always exist. Now, we can prove that the CT maps paraunitary matrices to PSH matrices and vice versa.

*Theorem 1:* The CT of a paraunitary matrix is a PSH matrix. Conversely, the CT of a PSH matrix is a paraunitary matrix.

*Proof:* See Appendix B. ■

## III. Orthogonal IIR Filter Banks

### A. Complete Characterization

By Theorem 1, the CT establishes a one-to-one mapping between paraunitary matrices and PSH matrices. For implementation purposes, we consider IIR filter banks with rational filters[3] only. In the polyphase domain, IIR filter banks lead to IIR matrices, entries of which are rational functions. Based on Theorem 1, we can obtain the complete characterization of orthogonal IIR filter banks with rational filters.

*Proposition 3:* The complete characterization of orthogonal IIR filter banks with rational filters in the Cayley domain is PSH matrices with rational entries.

*Proof:* By Theorem 1 and the fact that the CT of a rational matrix is still rational. ■

By Proposition 3, to design an orthogonal IIR filter bank, we can first design a PSH matrix in the Cayley domain, and then map it back to a paraunitary polyphase matrix.

This method simplifies our design problem, since the PSH condition amounts to *linear* constraints on the matrix entries, while the paraunitary condition amounts to nonlinear ones. Take the two-channel case as an example; for multiple-channel cases, the results are similar. Let $\mathbf{U}(z)$ be a $2 \times 2$ paraunitary matrix with

$$\mathbf{U}(z) = \begin{pmatrix} U_{00}(z) & U_{01}(z) \\ U_{10}(z) & U_{11}(z) \end{pmatrix}.$$

Then, the paraunitary condition $\mathbf{U}(z)\mathbf{U}^T(z^{-1}) = \mathbf{I}$ becomes

$$\begin{cases} U_{00}(z)U_{00}(z^{-1}) + U_{01}(z)U_{01}(z^{-1}) = 1 \\ U_{00}(z)U_{10}(z^{-1}) + U_{01}(z)U_{11}(z^{-1}) = 0 \\ U_{10}(z)U_{10}(z^{-1}) + U_{11}(z)U_{11}(z^{-1}) = 1. \end{cases} \quad (6)$$

Solving the system (6) involves solving three nonlinear equations with respect to the entries of $\mathbf{U}(z)$. In contrast, let $\mathbf{H}(z)$ be a $2 \times 2$ PSH matrix with

$$\mathbf{H}(z) = \begin{pmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{pmatrix}.$$

Then, the PSH condition $\mathbf{H}(z^{-1}) = -\mathbf{H}^T(z)$ becomes

$$\begin{cases} H_{00}(z^{-1}) = -H_{00}(z) \\ H_{11}(z^{-1}) = -H_{11}(z) \\ H_{01}(z^{-1}) = -H_{10}(z). \end{cases} \quad (7)$$

Here, solving the system (7) involves solving only three independent linear equations with respect to the entries of $\mathbf{H}(z)$. Moreover, these equations are decoupled. Specifically, we only need to design two antisymmetric filters, $H_{00}(z)$ and $H_{11}(z)$, independently and then choose one arbitrary filter $H_{10}(z)$ leading to $H_{01}(z)$ as in the last equation of (7). Then,

[3]A filter is said to be rational if its $z$ transform is a rational function.

the problem of designing $\mathbf{H}(\boldsymbol{z})$ converts to that of designing antisymmetric filters, which is a simple problem as shown by the following proposition.

*Proposition 4:* Suppose $W(\boldsymbol{z})$ is an IIR filter with a rational function: $W(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials. Then, $W(\boldsymbol{z}^{-1}) = -W(\boldsymbol{z})$ *if and only if*

$$A(\boldsymbol{z}^{-1}) = c\boldsymbol{z}^{\boldsymbol{m}}A(\boldsymbol{z}) \quad \text{and} \quad B(\boldsymbol{z}^{-1}) = -c\boldsymbol{z}^{\boldsymbol{m}}B(\boldsymbol{z})$$

where $\boldsymbol{m}$ is an arbitrary integer vector and $c = \pm 1$.

*Proof:* See Appendix E. ■

### *B. Design Examples of 2-D Quincunx IIR Orthogonal Filter Banks With Vanishing Moments*

In this subsection, we consider the design of 2-D quincunx orthogonal IIR filter banks as an illustration of the CT design method. It is straightforward to extend it to the the higher dimensional and more channels cases.

Quincunx sampling is density-2 sampling, leading to the two-channel case. Of all MD sampling patterns, the quincunx one is the most common. However, since the sampling is nonseparable, it offers challenges. For the quincunx sampling, its sampling matrix and the integer vectors in the fundamental parallelepiped in (5) can be written as

$$\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{l}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \boldsymbol{l}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From (5), the low-pass filter $G_0(\boldsymbol{z})$ of the quincunx filter bank can be written as

$$G_0(z_1, z_2) = U_{00}(z_1 z_2, z_1 z_2^{-1}) + z_1 U_{10}(z_1 z_2, z_1 z_2^{-1}). \quad (8)$$

In the context of wavelet design, the vanishing-moment condition plays an essential role. This condition requires the low-pass filter $G_0(z_1, z_2)$ to have $L$th-order zero derivatives at $\boldsymbol{z} = [-1, -1]^T$, i.e.

$$\left.\frac{\partial^n G_0(z_1, z_2)}{\partial z_1^i \partial z_2^{n-i}}\right|_{(-1,-1)} = 0$$
$$\text{for} \quad n = 0, 1, \ldots, L-1;\ i = 0, 1, \ldots, n \quad (9)$$

where the number of equations equals to the number of vanishing moments.

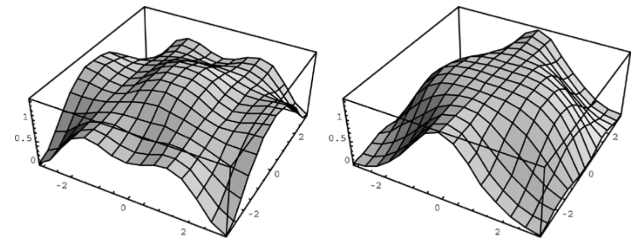
The design process is as follows. First, we parameterize a PSH matrix $\mathbf{H}(\boldsymbol{z})$ as shown in (7) with the help of Proposition 4. Second, we compute the CT of $\mathbf{H}(\boldsymbol{z})$ to get $\mathbf{U}(\boldsymbol{z})$ and obtain the low-pass filter from the polyphase matrix $\mathbf{U}(\boldsymbol{z})$ as in (8). Third, we impose the vanishing-moment condition on the low-pass filter.

To illustrate our design process, we design two orthogonal filters with the second-order vanishing moments. In terms of (9), the number of vanishing moments is 3, and, hence, the number of free variables in the parameterization for $\mathbf{H}(\boldsymbol{z})$ is also 3.

*Example 1:*

1) In this example, we choose $H_{00}(z_1, z_2)$ and $H_{11}(z_1, z_2)$ to be zeros, and parameterize $H_{01}(z_1, z_2)$ as

$$H_{01}(z_1, z_2) = \frac{1}{a_1 + a_2 z_1 + a_3 z_2}.$$



(a) (b)

Fig. 3. Magnitude frequency responses of two orthogonal low-pass filters with second-order vanishing moments, obtained by the CT. (a) Example 1. (b) Example 2.

2) Take the CT of $\mathbf{H}(\boldsymbol{z})$ to get $\mathbf{G}_p(\boldsymbol{z}) = \mathbf{U}(\boldsymbol{z})$ and impose the second-order vanishing-moment condition on the low-pass filter $G_0(\boldsymbol{z})$ as in **(9)**, which leads to

$$\begin{cases} a_2 = a_3 \\ a_1 = -2a_2 - 2a_3 \\ (a_1 + a_2 + a_3 - 1)^2 = 2. \end{cases}$$

3) Obtain the solutions

$$a_2 = a_3 = -\frac{1}{2} \pm \frac{1}{\sqrt{2}}, \quad a_1 = -4a_2.$$

The magnitude frequency response of the resulting low-pass filter in the quincunx filter bank is given in Fig. 3(a).

*Example 2:*

1) In this example, we choose $\mathbf{H}(z_1, z_2)$ to be FIR, and parameterize antisymmetric filters $H_{00}(z_1, z_2)$ and $H_{11}(z_1, z_2)$ as

$$H_{00}(z_1, z_2) = a_1(z_1 - z_1^{-1})$$
$$H_{11}(z_1, z_2) = a_2(z_2 - z_2^{-1})$$

and $H_{01}(z_1, z_2)$ as

$$H_{01}(z_1, z_2) = a_3.$$

2) Take the CT of $\mathbf{H}(\boldsymbol{z})$ to get $\mathbf{G}_p(\boldsymbol{z}) = \mathbf{U}(\boldsymbol{z})$ and impose the second-order vanishing-moment condition on the low-pass filter $G_0(\boldsymbol{z})$ as in **(9)**, which leads to

$$\begin{cases} a_2 = a_1 + a_3 \\ a_3^2 + 2a_3 - 1 = 0 \\ a_1(1 - a_3) + a_3(1 + a_3)a_2 = 0. \end{cases}$$

3) Obtain the solutions

$$a_3 = -1 \pm \sqrt{2}, \quad a_1 = \frac{-a_3}{2}, \quad a_2 = \frac{a_3}{2}.$$

The resulting low-pass filter in the quincunx filter bank has good diamond-like shape of the magnitude frequency response, as shown in Fig. 3(b).

## IV. Orthogonal FIR Filter Banks

In the polyphase domain, FIR filter banks lead to FIR matrices, entries of which are polynomials. To design orthogonal FIR filter banks, we need to design paraunitary FIR matrices.

In this section, we assume that $\mathbf{U}(\boldsymbol{z})$ is an $N \times N$ paraunitary FIR matrix. From (1), $\det \mathbf{U}(\boldsymbol{z}) \cdot \det \mathbf{U}^T(\boldsymbol{z}^{-1}) = 1$. Therefore, $\det \mathbf{U}(\boldsymbol{z})$ is an allpass FIR filter and, hence, a monomial [11]. Thus, we have

$$\det \mathbf{U}(\boldsymbol{z}) = c\boldsymbol{z}^{-\boldsymbol{k}}.$$

where $c = \pm 1$, and $\boldsymbol{k}$ is the McMillan degree of the orthogonal filter bank derived from $\mathbf{U}(\boldsymbol{z})$, which is defined as the minimum number of delay units for each dimension [18].

As mentioned in Section I, although $\mathbf{U}(\boldsymbol{z})$ is an FIR matrix, in general, $\mathbf{H}(\boldsymbol{z})$ is not because of the factor $\left(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\right)^{-1}$. Thus, we need to find a complete characterization of PSH matrices such that their CTs are FIR.

According to the Cramer's rule, we have

$$\begin{aligned}\mathbf{H}(\boldsymbol{z}) &= \left(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\right)^{-1}\left(\mathbf{I} - \mathbf{U}(\boldsymbol{z})\right)\\ &= \frac{\operatorname{adj}\left(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\right)\left(\mathbf{I} - \mathbf{U}(\boldsymbol{z})\right)}{\det\left(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\right)}\end{aligned} \tag{10}$$

where $\operatorname{adj} \mathbf{A}$ denotes the adjugate of $\mathbf{A}$. In other words, $\mathbf{H}(\boldsymbol{z})$ can be represented as the quotient of an FIR matrix and an FIR filter. Let $D(\boldsymbol{z})$ and $\mathbf{H}'(\boldsymbol{z})$ be the scaled denominator and numerator of (10), respectively

$$D(\boldsymbol{z}) \stackrel{\text{def}}{=} 2^{-N+1} \det\left(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\right) \tag{11}$$

$$\mathbf{H}'(\boldsymbol{z}) \stackrel{\text{def}}{=} 2^{-N+1} \operatorname{adj}\left(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\right)\left(\mathbf{I} - \mathbf{U}(\boldsymbol{z})\right) \tag{12}$$

where the scale factor $2^{-N+1}$ is introduced for later convenience. Then, $\mathbf{H}(\boldsymbol{z})$ can be expressed as

$$\mathbf{H}(\boldsymbol{z}) = \frac{\mathbf{H}'(\boldsymbol{z})}{D(\boldsymbol{z})}.$$

From Proposition 2, we have

$$\det\left(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\right) = 2^N \det\left(\mathbf{I} + \mathbf{H}(\boldsymbol{z})\right)^{-1}.$$

Substitute this into (11), and we get

$$D(\boldsymbol{z}) = 2 \det\left(\mathbf{I} + \mathbf{H}(\boldsymbol{z})\right)^{-1}. \tag{13}$$

Now, our task is to obtain a characterization of both $D(\boldsymbol{z})$ and $\mathbf{H}'(\boldsymbol{z})$, since they will characterize $\mathbf{H}(\boldsymbol{z})$.

*Lemma 1:* Suppose $\mathbf{U}(\boldsymbol{z})$ is a paraunitary FIR matrix of McMillan degree $\boldsymbol{k}$. Then, its CT $\mathbf{H}(\boldsymbol{z})$ can be written as $D(\boldsymbol{z})^{-1}\mathbf{H}'(\boldsymbol{z})$, where $D(\boldsymbol{z})$ is an FIR filter and $\mathbf{H}'(\boldsymbol{z})$ is an FIR matrix, and they satisfy the following conditions:

$$\begin{aligned}D(\boldsymbol{z}^{-1}) &= c\boldsymbol{z}^{\boldsymbol{k}} D(\boldsymbol{z})\\ \mathbf{H}'^T(\boldsymbol{z}^{-1}) &= -\,c\boldsymbol{z}^{\boldsymbol{k}}\mathbf{H}'(\boldsymbol{z})\\ 2\,D(\boldsymbol{z})^{N-1} &= \det\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right).\end{aligned}$$

Moreover, if $D(\boldsymbol{z})$ and $\mathbf{H}'(\boldsymbol{z})$ are coprime, then they are unique for each paraunitary FIR matrix $\mathbf{U}(\boldsymbol{z})$.

*Proof:* See Appendix C. ∎

Now, we formulate the complete characterization of paraunitary FIR matrices in the Cayley domain.

*Theorem 2:* The CT of a matrix $\mathbf{H}(\boldsymbol{z})$ is a paraunitary FIR matrix *if and only if* it can be written as $\mathbf{H}(\boldsymbol{z}) = D(\boldsymbol{z})^{-1}\mathbf{H}'(\boldsymbol{z})$, where $D(\boldsymbol{z})$ is an FIR filter and $\mathbf{H}'(\boldsymbol{z})$ is an FIR matrix, and they satisfy the following four conditions:

1) $D(\boldsymbol{z}^{-1}) = c\boldsymbol{z}^{\boldsymbol{k}} D(\boldsymbol{z})$;
2) $\mathbf{H}'^T(\boldsymbol{z}^{-1}) = -c\boldsymbol{z}^{\boldsymbol{k}}\mathbf{H}'(\boldsymbol{z})$;
3) $2\,D(\boldsymbol{z})^{N-1} = \det\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right)$;
4) $D(\boldsymbol{z})^{N-2}$ is a common factor of all minors of $D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})$.

Moreover, the CT of $\mathbf{H}(\boldsymbol{z})$ can be written as

$$\mathbf{U}(\boldsymbol{z}) = \frac{\operatorname{adj}\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right)}{D(\boldsymbol{z})^{N-2}} - \mathbf{I}. \tag{14}$$

*Proof:* By Lemma 1, we know that the first three conditions are necessary for the CT of $\mathbf{H}(\boldsymbol{z})$ to be a paraunitary FIR matrix. Furthermore, conditions 1) and 2) guarantee that $\mathbf{H}(\boldsymbol{z})$ is a PSH matrix, and, thus, its CT is a paraunitary matrix. Now, we only need to prove that condition 4) is the necessary and sufficient condition for the CT of $\mathbf{H}(\boldsymbol{z})$ to be FIR. By Proposition 2, the CT of $\mathbf{H}(\boldsymbol{z})$ is given by

$$\begin{aligned}\mathbf{U}(\boldsymbol{z}) &= 2\left(\mathbf{I} + \mathbf{H}(\boldsymbol{z})\right)^{-1} - \mathbf{I}\\ &= 2 - D(\boldsymbol{z})\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right)^{-1} - \mathbf{I}.\end{aligned}$$

From condition 3)

$$\begin{aligned}\mathbf{U}(\boldsymbol{z}) &= 2 - D(\boldsymbol{z})\frac{\operatorname{adj}\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right)}{2 - D(\boldsymbol{z})^{N-1}} - \mathbf{I}\\ &= \frac{\operatorname{adj}\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right)}{D(\boldsymbol{z})^{N-2}} - \mathbf{I}.\end{aligned}$$

To guarantee that $\mathbf{U}(\boldsymbol{z})$ is FIR, the necessary and sufficient condition is that $D(\boldsymbol{z})^{N-2}$ is a factor of $\operatorname{adj}\left(D(\boldsymbol{z})\,\mathbf{I}+\mathbf{H}'(\boldsymbol{z})\right)$. This completes the proof. ∎

Therefore, our problem of designing a paraunitary FIR matrix $\mathbf{U}(\boldsymbol{z})$ is converted to a problem of designing a PSH matrix $\mathbf{H}(\boldsymbol{z}) = D(\boldsymbol{z})^{-1}\mathbf{H}'(\boldsymbol{z})$, where $D(\boldsymbol{z})$ and $\mathbf{H}'(\boldsymbol{z})$ satisfy the conditions given in Theorem 2.

## V. Two-Channel Orthogonal FIR Filter Banks

### A. Complete Characterization

Among MD orthogonal filter banks, the two-channel ones are the simplest and most popular. In this case, $N = 2$ and condition 4) in Theorem 2 is always satisfied. Our goal here is to express $\mathbf{U}(\boldsymbol{z})$ directly as an FIR matrix using $D(\boldsymbol{z})$ and the terms from $\mathbf{H}'(\boldsymbol{z})$ so that we can impose further conditions on them (e.g., vanishing-moment conditions).

*Lemma 2:* Suppose $\mathbf{H}'(\boldsymbol{z})$ be a $2 \times 2$ FIR matrix with

$$\mathbf{H}'(\boldsymbol{z}) = \begin{pmatrix} H'_{00}(\boldsymbol{z}) & H'_{01}(\boldsymbol{z})\\ H'_{10}(\boldsymbol{z}) & H'_{11}(\boldsymbol{z})\end{pmatrix}.$$

Then, condition 3) in Theorem 2 amounts to the following two conditions:

$$\begin{cases} H'_{00}(\boldsymbol{z}) + H'_{11}(\boldsymbol{z}) = 1 - c\boldsymbol{z}^{-\boldsymbol{k}}\\ H'_{01}(\boldsymbol{z})H'_{01}(\boldsymbol{z}^{-1}) = c\boldsymbol{z}^{\boldsymbol{k}}\\ \qquad [2\,D(\boldsymbol{z}) - \left(D(\boldsymbol{z}) + H'_{00}(\boldsymbol{z})\right)\left(D(\boldsymbol{z}) + H'_{11}(\boldsymbol{z})\right)].\end{cases}$$

*Proof:* See Appendix D. ∎

Now we formulate a complete characterization of general $2 \times 2$ paraunitary FIR matrices.

*Theorem 3:* Any $2 \times 2$ MD paraunitary FIR matrix $\mathbf{U}(\boldsymbol{z})$ can be written as

$$\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} D(\boldsymbol{z}) + H'_{11}(\boldsymbol{z}) - 1 & -H'_{01}(\boldsymbol{z})\\ -H'_{10}(\boldsymbol{z}) & D(\boldsymbol{z}) + H'_{00}(\boldsymbol{z}) - 1\end{pmatrix} \tag{15}$$

where $D(\boldsymbol{z})$ is an FIR filter satisfying

$$D(\boldsymbol{z}^{-1}) = c\boldsymbol{z}^{\boldsymbol{k}}D(\boldsymbol{z})$$

and $\mathbf{H}'(\boldsymbol{z}) = \begin{pmatrix} H'_{00}(\boldsymbol{z}) & H'_{01}(\boldsymbol{z}) \\ H'_{10}(\boldsymbol{z}) & H'_{11}(\boldsymbol{z}) \end{pmatrix}$ is an FIR matrix satisfying

$$H'_{00}(\boldsymbol{z}^{-1}) = -c\boldsymbol{z}^{\boldsymbol{k}}H'_{00}(\boldsymbol{z}),$$
$$H'_{10}(\boldsymbol{z}^{-1}) = -c\boldsymbol{z}^{\boldsymbol{k}}H'_{01}(\boldsymbol{z}),$$
$$H'_{11}(\boldsymbol{z}) = 1 - c\boldsymbol{z}^{-\boldsymbol{k}} - H'_{00}(\boldsymbol{z})$$

and

$$H'_{01}(\boldsymbol{z})H'_{01}(\boldsymbol{z}^{-1}) = 2c\boldsymbol{z}^{\boldsymbol{k}}D(\boldsymbol{z}) \quad (16)$$
$$-c\boldsymbol{z}^{\boldsymbol{k}}\big(D(\boldsymbol{z}) + H'_{00}(\boldsymbol{z})\big)\big(D(\boldsymbol{z}) + H'_{11}(\boldsymbol{z})\big). \quad (17)$$

*Proof:* Let $\mathbf{H}(\boldsymbol{z})$ be the CT of $\mathbf{U}(\boldsymbol{z})$. Let $D(\boldsymbol{z})$ and $\mathbf{H}'(\boldsymbol{z})$ be defined as before. By Theorem 2 and Lemma 2, it is easy to verify that the CT of $\mathbf{H}(\boldsymbol{z}) = D(\boldsymbol{z})^{-1}\mathbf{H}'(\boldsymbol{z})$ is a paraunitary FIR matrix if, and only if, $D(\boldsymbol{z})$ and $\mathbf{H}'(\boldsymbol{z})$ satisfy the given conditions in this theorem.

It remains to show that the CT of $\mathbf{H}(\boldsymbol{z})$ satisfies (15). In the two-channel case, (14) becomes

$$\mathbf{U}(\boldsymbol{z}) = \operatorname{adj}\big(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\big) - \mathbf{I}$$

and $\operatorname{adj}\big(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\big)$ equals

$$\begin{pmatrix} D(\boldsymbol{z}) + H'_{11}(\boldsymbol{z}) & -H'_{01}(\boldsymbol{z}) \\ -H'_{10}(\boldsymbol{z}) & D(\boldsymbol{z}) + H'_{00}(\boldsymbol{z}) \end{pmatrix}.$$

This completes the proof. ■

In this characterization, the free parameters are the coefficients of the symmetric FIR filters $D(\boldsymbol{z})$ and $H'_{00}(\boldsymbol{z})$ only. Although the computation of $H'_{01}(\boldsymbol{z})$ in (16) amounts to a spectral factorization problem, it is easier since the size of filters here is half of that required in the design method of spectral factorizations by Smith and Barnwell [10]. In the actual design, we can parameterize the coefficients of the $D(\boldsymbol{z})$, $H'_{00}(\boldsymbol{z})$, and $H'_{10}(\boldsymbol{z})$, and then solve equations with respect to these coefficients imposed by (16), and possibly additional conditions. This design method is applicable for arbitrary dimensions. In the next subsection, we will detail the design process of 2-D orthogonal filter banks with vanishing moments.

### B. Design Examples of 2-D Quincunx Orthogonal FIR Filter Banks With Vanishing Moments

As in the orthogonal IIR case, the vanishing-moment condition plays an essential role in the context of wavelet design. Using (8) and (15), the low-pass filter $G_0(\boldsymbol{z})$ becomes

$$G_0(z_1,z_2) = (D(z_1z_2, z_1z_2^{-1}) + H'_{11}(z_1z_2, z_1z_2^{-1}) - 1)$$
$$-z_1H'_{10}(z_1z_2, z_1z_2^{-1}). \quad (18)$$

Therefore, the vanishing-moment condition imposes certain constraints on the derivatives of $D(\boldsymbol{z})$, $H'_{11}(\boldsymbol{z})$ and $H'_{10}(\boldsymbol{z})$ at $\boldsymbol{z} = [-1,-1]^T$. If we parameterize these three filters according to Theorem 3, then the vanishing-moment condition amounts to linear equations, which can be solved easily.

The design procedure is given as follows.

1) Parameterize FIR filters $D(z_1,z_2)$ and $H'_{11}(z_1,z_2)$ such that

$$D(z_1^{-1}, z_2^{-1}) = z_1^{k_1}z_2^{k_2}D(z_1,z_2), \quad (19)$$
$$H'_{11}(z_1^{-1}, z_2^{-1}) = -z_1^{k_1}z_2^{k_2}H'_{11}(z_1,z_2). \quad (20)$$

2) Parameterize the FIR filter $H'_{10}(z_1,z_2)$.
3) Compute the parameterization of the low-pass filter $G_0(z_1,z_2)$ using **(18)** and impose the $L$th-order vanishing moment as in **(9)**, resulting in linear equations.
4) Equation **(16)** in Theorem 3 becomes

$$H'_{10}(z_1,z_2)H'_{10}(z_1^{-1},z_2^{-1}) = 2z_1^{k_1}z_2^{k_2}D(z_1,z_2) - z_1^{k_1}z_2^{k_2}$$
$$\big(D(z_1,z_2)+H'_{00}(z_1,z_2)\big)\big(D(z_1,z_2)+H'_{11}(z_1,z_2)\big). \quad (21)$$

Comparing the coefficients of $\boldsymbol{z}$ results in quadratic equations.
5) Solve the set of all resulting equations from steps 3) and 4) together, and get all the filterbank filters from $\mathbf{U}(z_1,z_2)$ using **(15)**.

### C. Parameter Analysis

We want to examine the effect of the size of the filters on the number of equations to solve, and the maximum number of vanishing moments that can be imposed using the above design procedure.

1) The McMillan degree $[k_1,k_2]^T$ of an orthogonal filter bank is equal to the size of the filters. By **(19) and (20)**, the size of both $D(z_1,z_2)$ and $H'_{11}(z_1,z_2)$ are $(k_1+1)(k_2+1)$. By **(21)**, the size of $H'_{10}(z_1,z_2)$ is also $(k_1+1)(k_2+1)$.
2) Because of the symmetric conditions in **(19) and (20)**, the total number of parameters for $D(z_1,z_2)$ and $H'_{11}(z_1,z_2)$ is $(k_1 + 1)(k_2 + 1)$. Adding the number of parameters of $H'_{10}(z_1,z_2)$, the total number of unknowns is $2(k_1+1)(k_2+1)$.
3) Let $V$ be the number of vanishing moments. Then, step 3) leads to $V$ equations. In step 4), we get $\big((2k_1 + 1)(2k_2 + 1) + 1\big)/2$ independent equations from **(21)**. Therefore, to guarantee a solution (i.e., the number of equations must be less than or equal to the number of unknowns), the number of vanishing moments $V$ has to obey

$$V \le 2(k_1+1)(k_2+1) - \frac{(1+2k_1)(1+2k_2)+1}{2}$$
$$= k_1 + k_2 + 1. \quad (22)$$

### D. Design Examples

*Example 3:* In the first design example, we design orthogonal filters of minimum size, with the second-order vanishing

TABLE I
SIX SOLUTIONS YIELDING ORTHOGONAL FILTERS WITH SECOND-ORDER VANISHING MOMENTS

| No. | 1 | 2 | 3 |
|---|---|---|---|
| $a_1$ | $\frac{1}{4}(2 \pm \sqrt{2})$ | $\frac{1}{16}(8 \pm \sqrt{2} + \sqrt{6})$ | $\frac{1}{16}(8 \pm \sqrt{2} + \sqrt{6})$ |
| $a_2$ | 0 | $\frac{1}{16}(-\sqrt{6} \pm \sqrt{18})$ | $\frac{1}{16}(-\sqrt{6} \pm \sqrt{18})$ |
| $a_3$ | $\frac{1}{8}(4 \pm \sqrt{2} - \sqrt{6})$ | $\frac{1}{8}(4 \pm \sqrt{2})$ | $\frac{1}{8}(4 \pm \sqrt{2})$ |
| $a_4$ | 0 | $\frac{1}{8}\sqrt{6}$ | $-\frac{1}{8}\sqrt{6}$ |
| $a_5$ | $\frac{1}{8}(\pm\sqrt{2} - \sqrt{6})$ | $\frac{1}{16}(\mp\sqrt{2} - \sqrt{6})$ | $\frac{1}{16}(\mp\sqrt{2} - \sqrt{6})$ |

moments. According to (9), the second-order vanishing-moment condition (i.e., $L = 2$) specifies three equations or three vanishing moments. Therefore, using (22), a minimal McMillan degree is $[1, 1]^T$. The design procedure is given as follows.

1) Parameterize $D(z_1, z_2)$ and $H'_{11}(z_1, z_2)$ as

$$D(z_1, z_2) = a_1(1 + z_1^{-1}z_2^{-1}) + a_2(z_1^{-1} + z_2^{-1}),$$
$$H'_{11}(z_1, z_2) = a_3(1 - z_1^{-1}z_2^{-1}) + a_4(z_1^{-1} - z_2^{-1}).$$

2) Parameterize $H'_{10}(z_1, z_2)$ as

$$H'_{10}(z_1, z_2) = a_5 + a_6 z_1^{-1} + a_7 z_2^{-1} + a_8 z_1^{-1} z_2^{-1}.$$

3) Solve three linear equations as in **(9)** and we get

$$\begin{cases} a_6 = a_3 + a_4 - a_5 - \frac{1}{2} \\ a_7 = a_3 - a_4 - a_5 - \frac{1}{2} \\ a_8 = a_5 + 2a_1 + 2a_2 - 2a_3. \end{cases}$$

4) Comparing the coefficients of $z$ in **(21)**, which generates five quadratic equations. Solve these equations.

The resulting six solutions, which make up three pairs and each pair of filters are related by reversal, are given in Table I. The first pair leads to the degenerated Daubechies' D4 filter [7]. The second and third pairs are nonseparable filters same as those found by Kovačević and Vetterli [13] using the lattice structure.

*Example 4:* In the second example, we design orthogonal filters of minimum size, with the third-order vanishing moments, in which the number of vanishing moments is 6 in terms of (9). Therefore, using (22), a minimal McMillan degree in this case is $[3, 2]^T$. The algebraic solutions are difficult, but the numerical solutions are possible. There are totally eight independent[4] solutions, where two are degenerated Daubechies' D6 filters. The magnitude frequency responses of the rest six 2-D filters are shown in Fig. 4. The two filters in Fig. 4(a) are the same as those found by Kovačević and Vetterli [19] using the lattice structure. Note that the lattice structure is not a complete characterization in the MD case. In contrast, our design method finds all solutions owing to the complete characterization. Among the four new solutions given in Fig. 4(b), the top two have desirable diamond-like magnitude frequency response shape and the resulting 2-D filters $h[n_1, n_2]$ are given in Tables II and III.

[4]It is easy to verify that the reversal or modulation of an orthogonal FIR filter with the third-order vanishing moment is still orthogonal with the third-order vanishing moment.

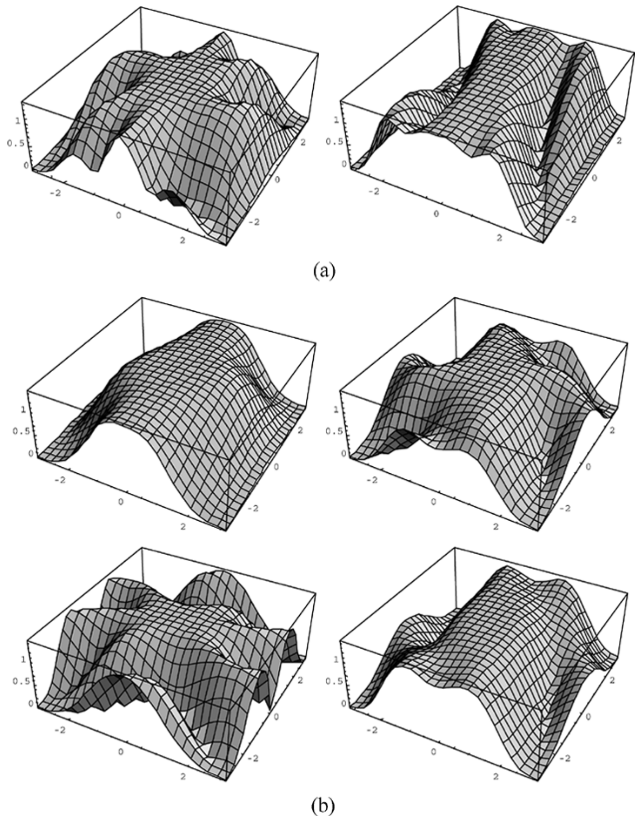


Fig. 4. Magnitude frequency responses of 24-tap orthogonal low-pass filters with third-order vanishing moments, obtained by the CT. (a) Two possible solutions, also found by Kovačević and Vetterli. (b) Four possible solutions. They are new filters.

TABLE II
COEFFICIENTS OF THE TOP LEFT FILTER IN FIG. 4(b) $h[n_1, n_2]$

| $n_1 \setminus n_2$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.075488 | 0 | 0 | 0 |
| 1 | 0 | 0.012489 | -0.040651 | -0.435382 | 0 | 0 |
| 2 | 0.000310 | -0.006588 | -0.206156 | -0.817362 | 0.028754 | 0 |
| 3 | -0.000163 | -0.005753 | 0.077692 | -0.255153 | 0.054036 | -0.001530 |
| 4 | 0 | 0.003061 | 0.086578 | 0.084412 | -0.011057 | -0.002914 |
| 5 | 0 | 0 | -0.046121 | 0.004306 | -0.020505 | 0 |
| 6 | 0 | 0 | 0 | 0.007996 | 0 | 0 |

TABLE III
COEFFICIENTS OF THE TOP RIGHT FILTER IN FIG. 4(b) $h[n_1, n_2]$

| $n_1 \setminus n_2$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.099592 | 0 | 0 | 0 |
| 1 | 0 | -0.162877 | -0.241812 | -0.096059 | 0 | 0 |
| 2 | -0.000530 | 0.396194 | 0.107243 | 0.233489 | 0.031684 | 0 |
| 3 | 0.000095 | 0.726965 | 0.112575 | -0.121339 | -0.077174 | 0.000010 |
| 4 | 0 | 0.299450 | 0.106067 | -0.066143 | 0.041963 | 0.000056 |
| 5 | 0 | 0 | 0.043673 | -0.025611 | 0.017251 | 0 |
| 6 | 0 | 0 | 0 | -0.010548 | 0 | 0 |

They can be used to construct wavelets bases via iterated filterbanks [13]. After the tenth iteration, they have smooth surface as shown in Fig. 5.

## VI. CONCLUSION

Designing MD orthogonal filter banks amounts to designing MD paraunitary matrices. The CT establishes a one-to-one mapping between MD paraunitary matrices and MD PSH matrices

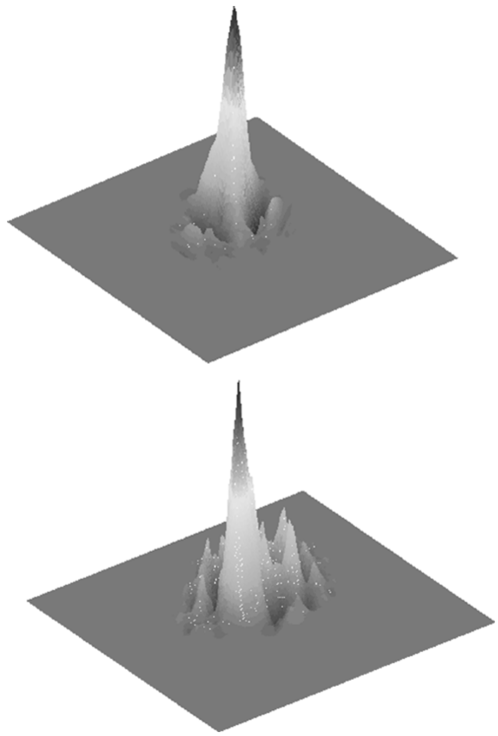


Fig. 5. Tenth iteration leading to quincunx wavelet bases of the top two filters in Fig. 4(b), obtained by the CT.

and converts nonlinear paraunitary condition to linear PSH condition. Based on this mapping, we find a uniform framework for the characterization of MD multichannel orthogonal filter banks.

The characterization of MD orthogonal IIR filter banks is simple: the linear space of PSH IIR matrices. In contrast, since the CT destroys the FIR property, the CT of an MD orthogonal FIR filter bank is in general not FIR. Instead, we find the characterization of a paraunitary FIR matrix in the Cayley domain is the quotient of an FIR matrix and an FIR filter with some constraints. In addition, we propose the design process to impose vanishing moments on orthogonal filters with illustrative examples. Our future work will be on the simplification of characterization for orthogonal FIR filter banks with more than two channels, and new filter designs for specific filter bank problems.

## APPENDIX A
## PROOF OF PROPOSITION 1

We prove it by induction.

1) When $N=1$, then $\Lambda = 1$, or $\Lambda=-1$. Obviously, either $1+\mathbf{U}(\boldsymbol{z})$ or $1-\mathbf{U}(\boldsymbol{z})$ must be nonsingular.
2) Suppose the proposition holds for $N=k$. Now, consider $N=k+1$. Express the $(k+1)\times(k+1)$ matrix $\mathbf{U}(\boldsymbol{z})$ as

$$\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} \mathbf{U}_k & \mathbf{U}_{1,k} \\ \mathbf{U}_{k,1} & u_{k,k} \end{pmatrix}.$$

By assumption, we can find a $k\times k$ diagonal matrix $\Lambda^*$ such that $\mathbf{I}+\Lambda^*\mathbf{U}_k$ is nonsingular. Let $\Lambda_1$ and $\Lambda_2$ be two $(k+1)\times(k+1)$ diagonal matrices such that

$$\Lambda_1 = \begin{pmatrix} \Lambda^* & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda_2 = \begin{pmatrix} \Lambda^* & 0 \\ 0 & -1 \end{pmatrix}.$$

We will show that either $\mathbf{I}+\Lambda_1\mathbf{U}(\boldsymbol{z})$ or $\mathbf{I}+\Lambda_2\mathbf{U}(\boldsymbol{z})$ must be nonsingular. We have

$$\mathbf{I}+\Lambda_1\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} \mathbf{I}_k+\Lambda^*\mathbf{U}_k & \Lambda^*\mathbf{U}_{1,k} \\ \mathbf{U}_{k,1} & 1+u_{k,k} \end{pmatrix}$$
$$\mathbf{I}+\Lambda_2\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} \mathbf{I}_k+\Lambda^*\mathbf{U}_k & \Lambda^*\mathbf{U}_{1,k} \\ -\mathbf{U}_{k,1} & 1-u_{k,k} \end{pmatrix}$$

where $\mathbf{I}_k$ is a $k\times k$ identity matrix.

Let $\mathbf{W}$ be a $(k+1)\times(k+1)$ matrix such that its first $k$ rows are the same as those of both $\mathbf{I}+\Lambda_1\mathbf{U}(\boldsymbol{z})$ and $\mathbf{I}+\Lambda_2\mathbf{U}(\boldsymbol{z})$, and its last row is the sum of the last rows of $\mathbf{I}+\Lambda_1\mathbf{U}(\boldsymbol{z})$ and $\mathbf{I}+\Lambda_2\mathbf{U}(\boldsymbol{z})$. Then

$$\det\mathbf{W} = \det(\mathbf{I}+\Lambda_1\mathbf{U}(\boldsymbol{z})) + \det(\mathbf{I}+\Lambda_2\mathbf{U}(\boldsymbol{z})) \tag{23}$$

and

$$\begin{aligned}\det\mathbf{W} &= \det\begin{pmatrix} \mathbf{I}_k+\Lambda^*\mathbf{U}_k & \Lambda^*\mathbf{U}_{1,k} \\ 0 & 2 \end{pmatrix} \\ &= 2\,\det(\mathbf{I}+\Lambda^*\mathbf{U}_k).\end{aligned}$$

By assumption, $\det\mathbf{W}\neq 0$, and, thus, from (23), either $\mathbf{I}+\Lambda_1\mathbf{U}(\boldsymbol{z})$ or $\mathbf{I}+\Lambda_2\mathbf{U}(\boldsymbol{z})$ must be nonsingular.

■

## APPENDIX B
## PROOF OF THEOREM 1

For the first part, suppose that $\mathbf{U}(\boldsymbol{z})$ is a paraunitary matrix and $\mathbf{H}(\boldsymbol{z})$ is the CT of $\mathbf{U}(\boldsymbol{z})$. Then, using (2)

$$\begin{aligned}\mathbf{H}^T(\boldsymbol{z}^{-1}) &= \big(\mathbf{I}-\mathbf{U}^T(\boldsymbol{z}^{-1})\big)\big(\mathbf{I}+\mathbf{U}^T(\boldsymbol{z}^{-1})\big)^{-1} \\ &= \big(\mathbf{U}(\boldsymbol{z})\mathbf{U}^T(\boldsymbol{z}^{-1}) - \mathbf{U}^T(\boldsymbol{z}^{-1})\big) \\ &\quad \big(\mathbf{U}(\boldsymbol{z})\mathbf{U}^T(\boldsymbol{z}^{-1}) + \mathbf{U}^T(\boldsymbol{z}^{-1})\big)^{-1} \\ &= \big(\mathbf{U}(\boldsymbol{z})-\mathbf{I}\big)\mathbf{U}^T(\boldsymbol{z}^{-1})\big(\mathbf{U}^T(\boldsymbol{z}^{-1})\big)^{-1} \\ &\quad \cdot \big(\mathbf{U}(\boldsymbol{z})+\mathbf{I}\big)^{-1} \\ &= -\big(\mathbf{I}-\mathbf{U}(\boldsymbol{z})\big)\big(\mathbf{I}+\mathbf{U}(\boldsymbol{z})\big)^{-1}.\end{aligned}$$

It can be easily checked that $\big(\mathbf{I}-\mathbf{U}(\boldsymbol{z})\big)$ and $\big(\mathbf{I}+\mathbf{U}(\boldsymbol{z})\big)^{-1}$ are commutable. Therefore

$$\begin{aligned}\mathbf{H}^T(\boldsymbol{z}^{-1}) &= -\big(\mathbf{I}+\mathbf{U}(\boldsymbol{z})\big)^{-1}\big(\mathbf{I}-\mathbf{U}(\boldsymbol{z})\big) \\ &= -\mathbf{H}(\boldsymbol{z}).\end{aligned}$$

For the second part, suppose that $\mathbf{H}(\boldsymbol{z})$ is a PSH matrix and $\mathbf{U}(\boldsymbol{z})$ is the CT of $\mathbf{H}(\boldsymbol{z})$. Then

$$\begin{aligned}\mathbf{U}^T(\boldsymbol{z}^{-1}) &= \big(\mathbf{I}-\mathbf{H}^T(\boldsymbol{z}^{-1})\big)\big(\mathbf{I}+\mathbf{H}^T(\boldsymbol{z}^{-1})\big)^{-1} \\ &= \big(\mathbf{I}+\mathbf{H}(\boldsymbol{z})\big)\big(\mathbf{I}-\mathbf{H}(\boldsymbol{z})\big)^{-1}.\end{aligned}$$

Similarly, $(\mathbf{I}+\mathbf{H}(z))$ and $(\mathbf{I}-\mathbf{H}(z))^{-1}$ are commutable. Therefore

$$\begin{aligned}\mathbf{U}(z)\mathbf{U}^T(z^{-1}) &= (\mathbf{I}+\mathbf{H}(z))^{-1}(\mathbf{I}-\mathbf{H}(z))\\ &\quad (\mathbf{I}-\mathbf{H}(z))^{-1}(\mathbf{I}+\mathbf{H}(z))\\ &= \mathbf{I}.\end{aligned}$$

■

## Appendix C
## Proof of Lemma 1

For the first part, define $D(z)$ and $\mathbf{H}'(z)$ as in (11) and (12), respectively. It is clear that both $D(z)$ and $\mathbf{H}'(z)$ are FIR. It remains to show the symmetric property of $D(z)$ and $\mathbf{H}'(z)$. By (13)

$$\begin{aligned}D(z^{-1})^{-1} &= 2^{-1}\det\left(\mathbf{I}+\mathbf{H}(z^{-1})\right)^T\\ &= 2^{-1}\det\left(\mathbf{I}-\mathbf{H}(z)\right)\end{aligned}$$

which leads to

$$\begin{aligned}D(z)D(z^{-1})^{-1} &= \det\left(\mathbf{I}+\mathbf{H}(z)\right)^{-1}\cdot\det\left(\mathbf{I}-\mathbf{H}(z)\right)\\ &= \det\left[\left(\mathbf{I}+\mathbf{H}(z)\right)^{-1}\left(\mathbf{I}-\mathbf{H}(z)\right)\right]\\ &= \det\mathbf{U}(z)\\ &= cz^{-\boldsymbol{k}}.\end{aligned}$$

Thus, $D(z^{-1}) = cz^{\boldsymbol{k}}D(z)$, and, hence

$$\begin{aligned}\mathbf{H}'^T(z^{-1}) &= D(z^{-1})\mathbf{H}^T(z^{-1})\\ &= -cz^{\boldsymbol{k}}D(z)\mathbf{H}(z)\\ &= -cz^{\boldsymbol{k}}\mathbf{H}'(z).\end{aligned}$$

By (13)

$$\begin{aligned}D(z) &= 2\det\left(\mathbf{I}+\mathbf{H}(z)\right)^{-1}\\ &= 2\det\left(\mathbf{I}+D(z)^{-1}\mathbf{H}'(z)\right)^{-1}\\ &= 2\det\left[D(z)\left(D(z)\,\mathbf{I}+\mathbf{H}'(z)\right)^{-1}\right]\\ &= 2\,D(z)^N\det\left(D(z)\,\mathbf{I}+\mathbf{H}'(z)\right)^{-1}.\end{aligned}$$

The second part is obvious since $\mathbf{H}(z) = D(z)^{-1}\mathbf{H}'(z)$.

## Appendix D
## Proof of Lemma 2

By condition 3) in Theorem 2

$$2 - D(z) = \det\left(D(z)\,\mathbf{I}+\mathbf{H}'(z)\right). \tag{24}$$

It is easy to verify for the two-channel case that

$$\det\left(D(z)\mathbf{I}+\mathbf{H}'(z)\right) = D^2(z) + D(z)\mathrm{tr}\,\mathbf{H}'(z) + \det\,\mathbf{H}'(z).$$

Therefore

$$D^2(z) + \det\,\mathbf{H}'(z) = 2 - D(z) - D(z)\mathrm{tr}\,\mathbf{H}'(z). \tag{25}$$

By replacing $z$ by $z^{-1}$ in (25), after some simple manipulation, we obtain

$$z^{2\boldsymbol{k}}\left(D^2(z) + \det\,\mathbf{H}'(z)\right) = 2cz^{\boldsymbol{k}}D(z) + z^{2\boldsymbol{k}}D(z)\mathrm{tr}\,\mathbf{H}'(z). \tag{26}$$

By combining (25) and (26) we obtain

$$\mathrm{tr}\,\mathbf{H}'(z) = 1 - cz^{-\boldsymbol{k}}.$$

This completes the proof for the first equation. The second equation follows from (24) after some straightforward manipulation. ■

## Appendix E
## Proof of Proposition 4

The proof of the sufficient condition is obvious. For the necessary condition, suppose $W(z^{-1}) = -W(z)$. Then, $A(z^{-1})B(z) = -A(z)B(z^{-1})$. Since $A(z)$ and $B(z)$ are coprime polynomials, there exists an FIR filter $R(z)$, such that

$$A(z^{-1}) = R(z)A(z). \tag{27}$$

By replacing $z$ by $z^{-1}$ in (27), we obtain

$$A(z) = R(z^{-1})A(z^{-1}). \tag{28}$$

By multiplying (27) and (28), we have

$$R(z)R(z^{-1}) = 1.$$

Therefore, $R(z)$ is an allpass FIR filter, i.e., $R(z) = cz^{\boldsymbol{m}}$, which completes the proof. ■

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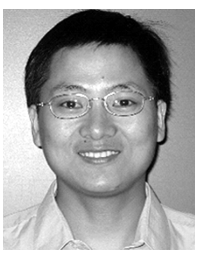

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