

Life Beyond Bases: The Advent of Frames (Part II)



While the first part of this article [73] presented most of the basic theoretical developments of frames, this part is more user friendly. It covers a large number of known frame families as well as those applications where frames made a difference. If you are familiar with the theory behind frames, you can just read this part and use it as necessary; all the relevant notation is given in Part I [73].

ALL IN THE FAMILY

We now consider particular frame families. The first two, harmonic tight frames and equiangular frames, are purely finite dimensional, while the rest are, in general, infinite dimensional. For some of the families, we will consider the unit-norm tight frame (UNTF) version and give the frame bound A yielding the redundancy of the frame family. We will denote by A_j the redundancy/frame bound at level j when iterated filter banks (FBs) are used.

HARMONIC TIGHT FRAMES AND VARIATIONS

Harmonic tight frames (HTF) are obtained by seeding from $\Psi = \text{DFT}_m$ (given in [73]), by deleting the last $(m - n)$ columns:

$$\varphi_i = \sqrt{\frac{m}{n}} \left(W_m^0, W_m^i, \dots, W_m^{i(n-1)} \right), \quad (1)$$

for $i = 0, \dots, m - 1$. Since obtained as an instance of the Naimark Theorem (given in [73]), this

is thus a Parseval tight frame (PTF), that is, $\Phi\Phi^* = I$. The simplest example of an HTF is the Mercedes-Benz (MB) frame given in [73, “The Mercedes-Benz Frame”]. In [26], the authors define a more general version of the HTF, called general harmonic frames. They also show that the HTFs are unique up to a permutation of the orthonormal basis (ONB) and that every general harmonic frame is unitarily equivalent to a simple variation of an HTF.

HTFs have a number of interesting properties: 1) For $m = n + 1$, all equal-norm tight frames (ENTFs) are unitarily equivalent to it; in other words, since we have HTFs for all n, m , we have all ENTFs for $m = n + 1$. 2) It is the only equal-norm PTF (ENPTF) such that its elements are generated by a group of unitary operators with one generator. 3) HTFs are maximally robust (MR) to erasures [54].

These frames have been generalized in an exhaustive work by Vale and Waldron [100], where the authors look at frames with symmetries. Some of these they term HTFs [their definition is more general than what is given in (1)] and are the result of the operation of a unitary U on a finite Abelian group G . When G is cyclic, the resulting frames are cyclic. In [26], the HTFs we showed above are with $U = I$ and generalized HTFs are with $U = D$ diagonal. These are cyclic in the parlance of [100]. An example of a cyclic frame are $(n + 1)$ vertices of a regular simplex in \mathbb{R}^n . There exist HTFs which are not cyclic.

Similar ideas have appeared in the work by Eldar and Bölcskei [46] under the name geometrically uniform (GU) frames, frames defined over a finite Abelian group of unitary matrices both with a single generator as well as multiple generators. The authors also consider constructions of such frames from given frames, closest in the least-squares sense, a sort of a Gram-Schmidt procedure for GU frames.

GRASSMANIAN PACKINGS AND EQUIANGULAR FRAMES

Equiangular (referring to $|\langle\varphi_i, \varphi_j\rangle| = \text{const.}$) frame families have become popular recently due to their use in quantum computing. In that terminology, a rank-1 measurement is represented by a positive operator valued measure (POVM). Each rank-1 POVM is a tight frame.

The first family is symmetric informationally complete POVMs (SIC-POVMs) [86]. A SIC-POVM is a family Φ of $m = n^2$ vectors in \mathbb{C}^n such that

$$|\langle\varphi_i, \varphi_j\rangle|^2 = \frac{1}{n+1} \quad (2)$$

holds for all $i, j, i \neq j$. At this point, it is not known whether SIC-POVMs exist for all finite dimensions.

The second family is mutually unbiased bases (MUBs). A MUB is a family Φ of $n + 1$ ONBs in a Hilbert space of dimension n (for instance, \mathbb{C}^n) such that for any two different bases B_I, B_J and any vectors $\varphi_i \in B_I$ and $\varphi_j \in B_J$, we have

$$|\langle\varphi_i, \varphi_j\rangle|^2 = \frac{1}{n}. \quad (3)$$

Both harmonic tight frames and equiangular frames have strong connections to Grassmanian frames. In a comprehensive paper [96], Strohmer and Heath discuss those frames and their connection to Grassmanian packings, spherical codes, graph theory, Welch Bound sequences (see also [62]). These frames are of unit norm (not a necessary restriction) and minimize the maximum correlation $|\langle\varphi_i, \varphi_j\rangle|$ among all frames. The problem arises from looking at overcomplete systems closest to orthonormal bases (which have minimum correlation). A simple example is an HTF in \mathbb{H}^n . Theorem 2.3 in [96] states that, given a frame Φ :

$$\min_{\Phi} (\max_{(\varphi_i, \varphi_j)} |\langle\varphi_i, \varphi_j\rangle|) \geq \sqrt{\frac{m-n}{n(m-1)}}. \quad (4)$$

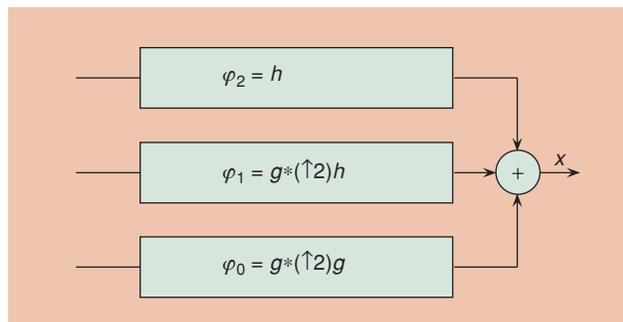
The equality in (4) is achieved if and only if Φ is equiangular and tight. In particular, for $\mathbb{H} = \mathbb{R}$, equality is possible only for $m \leq n(n+1)/2$, while for $\mathbb{H} = \mathbb{C}$, equality is possible only for $m \leq n^2$. Note that the above inequality is exactly the one Welch proved in [105] and which later led to what is today commonly referred to as the Welch’s Bound given in (7) by minimizing interuser interference in a code-division multiple access (CDMA) system [82] (see the discussion on the Welch’s Bound). In a more recent work, Xia et al. [107] constructed some new frames meeting the original Welch’s Bound (7).

These frames coincide with some optimal packings in Grassmanian spaces [32], spherical codes [33], equiangular lines [78], and many others. The equiangular lines are equivalent to the SIC-POVMs we discussed above.

THE ALGORITHME À TROUS

The *algorithme à trous* is a fast implementation of the dyadic (continuous) wavelet transform. It was first introduced by Holschneider, Kronland-Martinet, Morlet, and Tchamitchian in 1989 [63]. The transform is implemented via a biorthogonal, nondownsampling FB. An example for $j = 2$ levels is given in Figure 1 (this is essentially the same as the 2-level discrete wavelet transform (DWT) as in Part I [73, Figure 5], with samplers removed).

Let g and h be the filters used in this FB. At level i we will have equivalent upsampling by 2^i which means that the filter moved across the upsampler will be upsampled by 2^i , inserting $(2^i - 1)$ zeros between every two samples and thus creating holes (*trou* means hole in French).



[FIG1] The synthesis part of the FB implementing the *à trous* algorithm. The analysis part is analogous. This is equivalent to [73, Figure 5], with sampling removed.

Figure 2(d) shows the sampling grid for the à trous algorithm. It is clear from the figure, that this scheme is completely redundant, as all the points exist. This is in contrast to a completely nonredundant scheme such as the DWT, given in the top plot of the figure. In fact, the redundancy (or frame bound) of this algorithm grows exponentially since $A_1 = 2, A_2 = 4, \dots, A_j = 2^j, \dots$ (note that here we use a two-channel FB and that A_j is the frame bound when we use j levels), and the total redundancy for j levels is 2^{j+1} . This growing redundancy is the price we pay for shift invariance as well as the simplicity of the algorithm. The two-dimensional (2-D) version of the algorithm is obtained by extending the one-dimensional (1-D) version in a separable manner.

GABOR AND COSINE-MODULATED FRAMES

The idea behind this class of frames, consisting of many families, dates back to Gabor [53] and the insight of constructing bases by modulation of a single prototype function. Gabor originally used complex modulation, and thus, all those families with complex modulation are termed Gabor frames. Other types of modulation are possible, such as cosine modulation, and again, all those families with cosine modulation are termed cosine-modulated frames. Cosine-modulated bases are also often called Wilson bases. The connection between these two classes is deep as there exists a general decomposition of the frame operator corresponding to a cosine-modulated FB as the sum of the frame operator of the underlying Gabor frame (with the same prototype function and twice the redundancy) and an additional operator, which vanishes if the generator satisfies certain symmetry properties. While this decomposition has first been used by Auscher in the context of Wilson bases [4], it is valid more generally. Both of these classes can be seen as general oversampled FBs with m channels and sampling by n (see [73, Figure 7]).

GABOR FRAMES

A Gabor frame is $\Phi = \{\varphi_i\}_{i=0}^{m-1}$, with

$$\varphi_{i,k} = W_m^{-ik} \varphi_{0,k}, \quad (5)$$

where index $i = 0, \dots, m-1$ refers to the number of frame elements, $k \in \mathbb{Z}$ is the discrete-time index, W_m is the m th root of unity and φ_0 is the prototype frame function. Comparing (5) with (1), we see that for filter length $l = n$ and $\varphi_{0,k} = 1, k = 0$ and 0 otherwise, the Gabor system is equivalent to a HTF frame. Thus, it is sometimes called the oversampled discrete Fourier transform (DFT) frame.

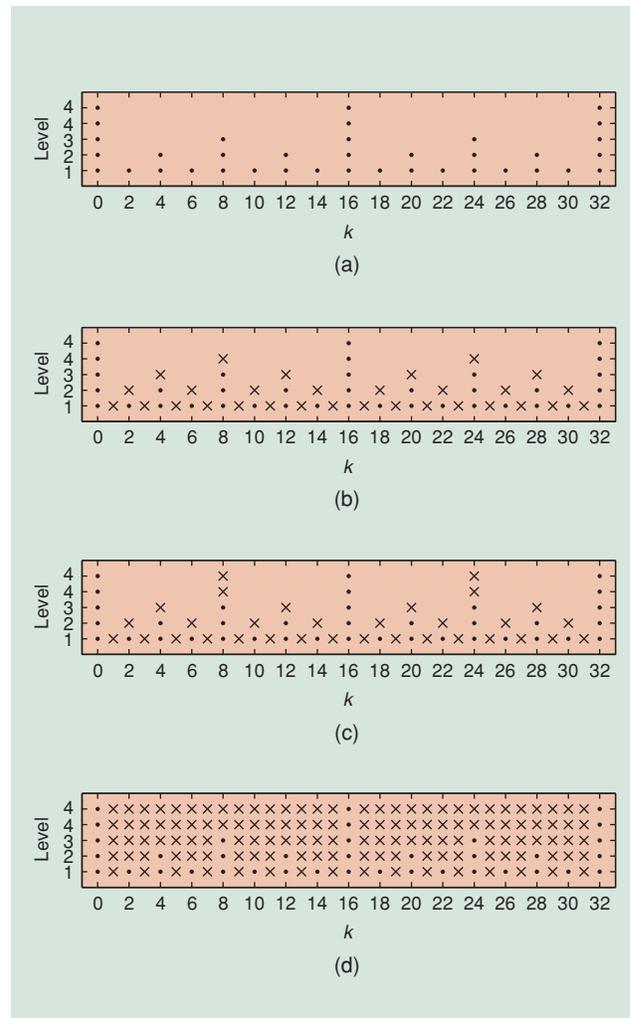
For the critically sampled case, one cannot have Gabor bases with good time and frequency localization at the same time (this is similar in spirit to the Balian-Low theorem which holds for $\mathcal{L}^2(\mathbb{R})$ [37]); this prompted the development of oversampled (redundant) Gabor systems (frames). They are known under various names: oversampled DFT FBs, complex-modulated FBs, short-time Fourier FBs and Gabor FBs and have been studied in [17], [18], [20], [35], [50], (see also [95]

and references within). More recent work includes [77], where the authors study finite-dimensional Gabor systems and show a family in \mathbb{C}^n , with $m = n^2$ vectors, which allows for $n^2 - n$ erasures, where n is prime (see “Robust Transmissions” section for discussion of erasures). In [74], new classes of Gabor ENTFs are shown, which are also MR.

COSINE-MODULATED FRAMES

The other kind of modulation, cosine, was used with great success within critically sampled FBs due to efficient implementation algorithms. Its oversampled version was introduced in [18], where the authors define the frame elements as:

$$\varphi_{i,k} = \sqrt{2} \cos\left(\frac{(i+1/2)\pi}{m} + \alpha_i\right) \varphi_{0,k}, \quad (6)$$



[FIG2] Sampling grids corresponding to time-frequency tilings of (a) DWT (nonredundant), (b) DD-DWT/Laplacian pyramid, (c) DT-CWT/PSDWT/PDWT, and (d) à trous family (completely redundant). Black dots correspond to the nonredundant (DWT-like) sampling grid. Crosses denote redundant points. Note that the last two ticks on the y-axis represent level 4 for the highpass and lowpass channels, respectively.

where index $i = 0, \dots, m - 1$ refers to the number of frame elements, $k \in \mathbb{Z}$ is the discrete-time index and φ_0 is the prototype frame function. The so-called odd-stacked cosine modulated FBs are defined in (6); even-stacked ones exist as well.

Cosine-modulated FBs do not suffer from time-frequency localization problems, given by a general result stating that the generating window of an orthogonal cosine modulated FB can be obtained by constructing a tight complex FB with oversampling factor 2 while making sure the window function satisfies a certain symmetry property (for more details, see [18]). Since we can get well-localized Gabor frames for redundancy 2, this also shows that we can get well-localized cosine-modulated FBs.

HARMONIC TIGHT FRAMES ARE A REDUNDANT COUNTERPART OF THE DISCRETE FOURIER TRANSFORM.

THE DUAL-TREE COMPLEX WAVELET TRANSFORM

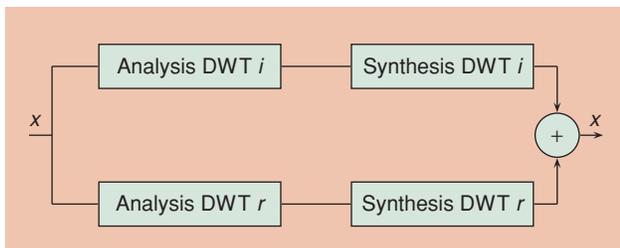
The dual-tree complex wavelet transform (DT-CWT) was first introduced by Kingsbury in 1998 [69]–[71]. The basic idea is to have two DWT trees working in parallel. One tree represents the real part of the complex transform while the second tree represents the imaginary part. That is, when the DT-CWT is applied to a real signal, the output of the first tree is the real part of the complex transform whereas the output of the second tree is its imaginary part. Shown in Figure 3 is the synthesis FB for the DT-CWT. Each tree uses a different pair of lowpass and highpass filters. These filters are designed so that they satisfy the perfect reconstruction condition.

Let Φ_r and Φ_i be the square matrices representing each of the DWTs in the DT-CWT. Then,

$$\Phi = \frac{1}{\sqrt{2}} (\Phi_r \ \Phi_i),$$

is a rectangular matrix, and thus a frame, representing the DT-CWT. The indices r and i stem from real and imaginary. The right inverse of Φ is the analysis FB (analysis operator) and is given by $\tilde{\Phi}^* = 1/\sqrt{2} ((\Phi_r)^{-1}(\Phi_i)^{-1})^T$. If Φ_r and Φ_i are unitary matrices, then $\tilde{\Phi} = \Phi$, $\Phi\Phi^* = I$, and Φ is a PTF.

Because the two DWT trees used in the DT-CWT are fully downsampled, the redundancy is only 2 for the 1-D case (it is 2^d for the d-dimensional case). We can see that in Figure 2(c),



[FIG3] The FB implementing the CWT. The two branches have two different two-channel FBs as in [73, Figure 3]. The analysis part is analogous.

where the redundancy at each level is twice that of the DWT, that is $A_1 = A_2 = \dots A_j = 2$. Unlike the à trous algorithm, however, here the redundancy is independent of the number of levels used in the transform.

When the two DWTs used are orthonormal, the DT-CWT is a tight frame. The DT-CWT overcomes one of the main drawbacks of the DWT: shift variance. Since the DT-CWT contains two fully downsampled DWTs which satisfy the half-sample delay condition (see below), aliasing due to downsampling can be largely eliminated and the transform becomes nearly shift invariant. The advantage the DT-CWT has over other complex transforms is that it has a fast invertible implementation and moreover, when the signal is real valued, the real and imaginary parts of its transform coefficients can be computed and stored separately.

As mentioned previously, the pairs of filters (h_r, g_r) and (h_i, g_i) of each DWT have to satisfy the perfect reconstruction condition. In addition, the filters have to be FIR and satisfy the so-called half sample delay condition, which implies that all of the filters have to be designed simultaneously. From this condition it also follows that the two highpass filters form an approximate Hilbert transform pair, and it thus makes sense to regard the outputs of the two trees as the real and imaginary parts of complex functions. Different design solutions exist, amongst them the linear phase biorthogonal one and the quarter-shift one [71], [91]. Moreover, we can use different-flavor trees to implement the DT-CWT. For example, it is possible to use a different pair of filters at each level, or alternate filters between the trees at each stage except for the first one.

In 2-D (or mD), the DT-CWT possesses directional selectivity allowing us to capture edge or curve information, a property clearly absent from the usual separable DWT. In the real case, orientation selectivity is simply achieved by using two real separable 2-D DWTs in parallel. Two pairs of filters are used to implement each DWT. These two transforms produce six subbands, three pairs of subbands from the same space-frequency region. By taking the sums and differences of each pair, one obtains the oriented wavelet transform.

The near shift invariance and orientation selectivity properties of the DT-CWT open up a window into a wide range of applications, among them denoising, motion estimation, image segmentation as well as building feature, texture and object detectors for images (see “Other Applications” section and references therein).

DOUBLE-DENSITY FRAMES AND VARIATIONS

The DT-CWT appears to be the most notable among the oversampled FB transforms. It is joined by a host of others: In particular, Selesnick in [89] introduces the double-density DWT (DD-DWT), which can approximately be implemented using a three-channel FB with sampling by 2 as in [73, Figure 6]. The

filters in the analysis bank are time-reversed versions of those in the synthesis bank. The redundancy of this FB tends towards 2 when iterated on the channel with φ_1 [73, Figure 6]. Actually, we have that $A_1 = (3/2), A_2 = (7/4), \dots, A_\infty = 2$ [see Figure 2(b)]. Like the DT-CWT, the DD-DWT is nearly shift invariant when compared to the à trous construction. In [90], Selesnick introduces the combination of the DD-DWT and the DT-CWT which he calls double-density, DT-CWT (DD-DT-CWT). This transform can be seen as the one in Figure 3 (DT-CWT), with individual FBs being overcomplete ones given in [73, Figure 6] (DD-DWT). In [1], Abdelnour and Selesnick introduce symmetric, nearly shift-invariant FBs implementing tight frames. These FBs have four filters in two couples, obtained from each other by modulation. Sampling is by 2 and thus the total redundancy is 2.

Another variation on a theme is the power-shiftable DWT (PSDWT) [92] or partial DWT (PDWT) [94], which removes samplers at the first level but leaves them at all other levels (see Figure 4). The sampling grid of the PSDWT/PDWT is shown in Figure 2(c). We see that it has redundancy $A_j = 2$ at each level (similarly to the CWT). The PSDWT/PDWT achieves near shift invariance.

Bradley in [22] introduces overcomplete DWT (OC-DWT), the DWT with critical sampling for the first k levels followed by à trous for the last $j - k$ levels. The OC-DWT becomes the à trous algorithm when $k = 0$ or the DWT when $k = j$.

MULTIDIMENSIONAL FRAMES

Apart from obvious, tensor-like, constructions (separate application of 1-D methods in each dimension) of multidimensional (mD) frames, we are interested in true mD solutions. The oldest mD frame seems to be the steerable pyramid introduced by Simoncelli, Freeman, Adelson and Heeger in 1992 [92], following on the previous work by Burt and Adelson on pyramid coding (see “Pyramid Coding” and [23]). The steerable pyramid possesses many nice properties, such as joint space-frequency localization, approximate shift invariance, approximate tightness, oriented kernels, approximate rotation invariance and a redundancy factor of $4j/3$, where j is the number of orientation subbands. The transform is implemented by a first stage of lowpass/highpass filtering followed by oriented bandpass filters in the lowpass branch plus another lowpass filter in the same branch followed by downsampling. An excellent overview of the steerable pyramid and its applications is given on Simoncelli’s web page [88].

Another beautiful example is the recent work of Do and Vetterli on contourlets [34], [42]. This work was motivated by the need to construct efficient and sparse representations of intrinsic geometric structure of information within an image. The authors combine the ideas of pyramid coding (see “Pyramid Coding”) and pyramid FBs [41] with directional processing, to obtain contourlets, expansions capturing contour segments. The transform is a frame composed of a pyramid FB

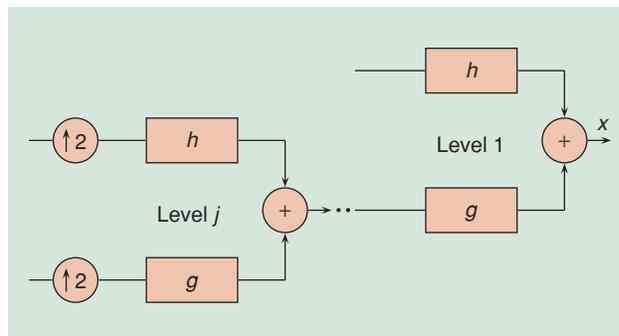
and a directional FB. Thus, first a wavelet-like method is used for edge detection (pyramid) followed by local directional transform for contour segment detection. It is almost critically sampled, with redundancy of 1.33. It draws on the ideas of a pyramidal directional FBs (PDFBs) which is a PTF when all the filters used are orthogonal (see Figure 5).

Some other examples include [79] where the authors build both critically sampled and nonsampled (à trous like) 2-D DWT. It is obtained by a separable 2-D DWT producing 4 subbands. The lowest subband is left as is, while the three higher ones are split into two subbands each using a quincunx FB (checkerboard sampling). The resulting FB possesses good directionality with low redundancy. Many “-lets” are also multidimensional frames, such as curvelets [24], [25] and shearlets [75]. As the name implies, curvelets are used to approximate curved singularities in an efficient manner [24], [25]. As opposed to wavelets which use dilation and translation, shearlets use dilation, shear transformation and translation, and possess useful properties such as directionality, elongated shapes and many others [75].

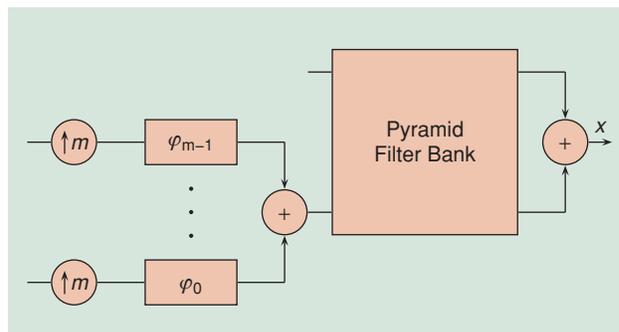
DISCUSSION AND NOTES

While in this section we aimed to present a comprehensive overview of frame families implementable by FBs, omissions are

**THE ALGORITHME À TROUS
IS THE MOST REDUNDANT
FILTER BANK FRAME.**



[FIG4] The synthesis part of the FB implementing the power-shiftable DWT. The samplers are omitted at the first level but exist at all other levels. The analysis part is analogous.



[FIG5] The synthesis part of the pyramid directional FB. The pyramid FB is given in “Pyramid Coding.” The scheme can be iterated and the analysis part is analogous.

probable. We note here some other developments, which, while not necessarily yet in the realm of FB frames, are related to them nevertheless.

For example, Casazza and Leonhard keep a tab on all equal-norm Parseval frames in [27]. In [6], [7], [56], the authors introduce the notion of localized frames, as an important new direction in frame theory, with possible FB instantiations in the future.

APPLICATIONS

We now present a glimpse at application domains where frames have been used with success. As with the previous material, we make no attempt to be exhaustive; we merely give a representative sample. These applications illustrate which basic properties of frames have found use in the real world. In some of these, frames have been used deliberately; by considering the requirements posed by applications, frames emerged as a natural choice. In others, only later have we become aware that the tools used were actually frames.

SOURCE CODING

An area where frames were immediately recognized as natural signal expansions was that of source coding. The success was motivated by the fact that frames show resilience to additive noise as well as numerical stability of reconstruction [37]. We start by illustrating resilience to noise. Intuitively, if a certain amount of noise is present, distributed over the transform coefficients (inner products), then it stands to reason that when there are m coefficients (frame) as opposed to n ($m > n$, bases), it is easier to deal with that lower level of noise per coefficient.

Example: We go back to our MB frame and consider its UNTF version given in [73, “The Mercedes-Benz Frame”]. Suppose we perturb our frame coefficients by adding white noise w_i to the channel i , where $E[w_i] = 0$, $E[w_i w_k] = \sigma^2 \delta_{ik}$ for $i, k = 1, 2, 3$. We can now find the error of the reconstruction,

$$\begin{aligned} x - \hat{x} &= \frac{2}{3} \sum_{i=1}^3 \langle \varphi_i, x \rangle \varphi_i - \frac{2}{3} \sum_{i=1}^3 (\langle \varphi_i, x \rangle + w_i) \varphi_i \\ &= -\frac{2}{3} \sum_{i=1}^3 w_i \varphi_i. \end{aligned}$$

Then the averaged mean-squared error per component is

$$\begin{aligned} \text{MSE} &= \frac{1}{2} E \|x - \hat{x}\|^2 = \frac{1}{2} E \left\| \frac{2}{3} \sum_{i=1}^3 w_i \varphi_i \right\|^2 \\ &= \frac{1}{2} \sigma^2 \frac{4}{9} \sum_{i=1}^3 \|\varphi_i\|^2 = \frac{2}{3} \sigma^2, \end{aligned}$$

since all the frame vectors have norm 1. Compare this to the same MSE obtained with an ONB: σ^2 . In other words:

$$\text{MSE}_{ONB} = \frac{3}{2} \text{MSE}_{MB},$$

that is, the amount of error per component has been reduced using a frame. \square

Frames are thus generally considered to be robust under additive noise [12], [37], [83]. While additive noise models for quantization error can be somewhat misleading as shown in [36], we use this example because it is simple and carries some intuition. Other works in the area include [11], [36], [38], [55], [58], and have been used with success in the context of sigma-delta quantization [19], [38], [57], and oversampled A/D conversion.

DENOISING

Denosing with wavelets can be traced back to the work by Weaver et al. [104] (and even earlier to Witkin [106]), and was later on popularized by Donoho and Johnstone [43], [44]. Even then, sophisticated use of overcomplete expansions showed excellent results, and thus one of the first works on denosing with frames is [108], where the authors combined the overcomplete expansion with a variation of the technique from [81] to reconstruct the image from its wavelet maxima.

More recent works include cycle spinning introduced by Coifman and Donoho [31]. They suggest that when using a j -stage wavelet transform, one can take advantage of the fact that there are effectively 2^j different wavelet bases, each one corresponding to one of the 2^j shifts. Thus one can denoise in each of those 2^j wavelet bases and then average the result. Even though errors of individual estimates are generally positively correlated, one gets an advantage from averaging the estimators. Another effect of this is that the shift-varying basis gives way to a shift-invariant frame (collection of bases). In [45], Dragotti et al. construct separable multidimensional directional frames for image denosing. The algorithm is in spirit similar to cycle spinning.

Ishwar and Moulin take a slightly different approach to develop a general framework for image modeling and estimation by fusing deterministic and statistical information from subband coefficients in multiple wavelet bases using maximum-entropy and set-theoretic ideas [64]–[67]. For instance, in [67] natural images are modeled as having sparse representations simultaneously in multiple orthonormal wavelet bases. Closed convex confidence tubes are designed around the wavelet coefficients of sparse initial estimates in multiple wavelet bases (frames). A POCS algorithm is then used to arrive at a globally consistent sparse signal estimate. Denosing and restoration algorithms based on these image models produced visually sharper estimates with about 1-2 dB PSNR gains over competitive denosing algorithms such as the spatially adaptive Wiener filter.

Some other works include that by Fletcher et al. [51], where the authors analyze denosing by sparse approximation with

THE STEERABLE PYRAMID IS ONE OF THE OLDEST FRAMES.

frames. The known a priori information about the signal x is that it has known sparsity k , that is, it can be represented via k nonzero frame coefficients (with respect to a given frame Φ). Then, after having been corrupted by noise yielding \hat{X} , the signal can be estimated by finding the best sparse approximation of \hat{X} . This work is essentially an attempt to understand how large a frame should be for denoising with a frame to be effective. In [30], the authors use the shift-invariant properties of the DT-CWT to provide better persistence across scales within the Hidden Markov tree, and hence better denoising performance, while in [84], the steerable pyramid is used (see “Multidimensional Frames” section). An example of denoising by frames is given in Figure 6 (courtesy of Vivek Goyal).

THE SUCCESS OF WAVELETS IN DENOISING IS FOLLOWED BY EVEN GREATER SUCCESS OF DENOISING WITH FRAMES.

ROBUST TRANSMISSION

Another application where frames found a natural home was that of robust transmission in communications. It was pioneered by Goyal, Kovačević and Kelner in [54], and was followed by works in [13]–[16], [26], [62], [74], [85], [96], [101]. The problem was that of creating multiple descriptions of the source so that when transmitted, and in the presence of losses, the source could be reconstructed based on received material. This clearly means that some amount of redundancy needs to be present in the system, since, if not, the loss of even one description would be irreversible.

In the initial work, the \mathbb{R}^n -valued source vector x is represented through a frame expansion with frame operator Φ^* , yielding $X = \Phi^*x \in \mathbb{R}^m$. The scalar quantization of the frame expansion coefficients gives \hat{X} lying in a discrete subset of \mathbb{R}^m . One abstracts the effect of the network to be the erasure of some components of \hat{X} . This implies that the components of \hat{X} are placed in more than one packet, for otherwise all of \hat{X} could be lost at once. If they are placed in m separate packets, then any subset of the components of \hat{X} may be received; otherwise only certain subsets are possible. The authors assume that linear reconstruction is

used, that is, the dual frame is used to reconstruct. The authors model the noise as additive $\eta = \hat{X} - X$ as in “Source Coding” with the assumptions that every noise component is of zero mean and variance σ^2 and that they are uncorrelated. The canonical dual frame operator (see [73]) is used as it minimizes the error of reconstruction. Losses in the network are modeled as erasures of a subset of quantized frame coefficients; to the decoder, it appears as if a quantized frame expansion were computed with the frame missing the elements which produced the erased ones, and thus, assuming it is a frame, a dual frame can be found. As a result, the authors concentrated on questions such as which deletions still leave a frame, which are the frames remaining frames under deletions of any subset of elements (up to $m - n$), etc. An example of this discussion is given for the MB frame in [73, “The Mercedes-Benz Frame”].

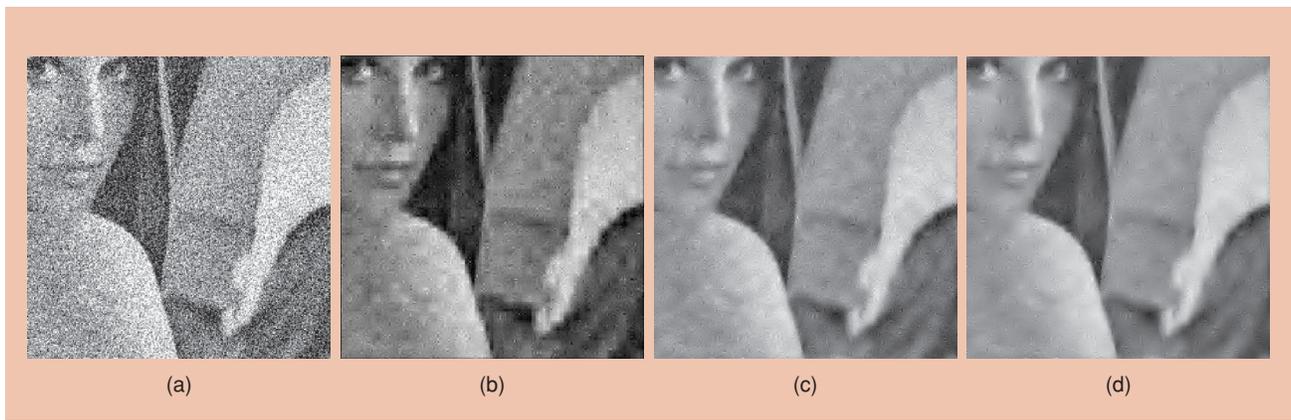
CDMA SYSTEMS

The use of frames in CDMA systems dates back to the work of Massey and Mittelholzer [82] on Welch’s Bound and sequence sets for CDMA systems.

In a CDMA system, there are m users who share the available spectrum. The sharing is achieved by scrambling m -dimensional user vectors into smaller, n -dimensional vectors. In terms of frame theory, this scrambling corresponds to the application of a synthesis operator corresponding to m distinct n -dimensional signature vectors φ_i of norm \sqrt{n} . Noise-corrupted versions of these synthesized vectors arrive at a receiver, where the signature vectors are used to help extract the original user vectors. The variance of the interuser interference for user i is:

$$\sigma_i^2 = \sum_{j=1}^m |\langle \varphi_i, \varphi_j \rangle|^2 - n^2,$$

leading to the total interuser interference:



[FIG6] Denoising with frames. (a) Lena with 34.0 dB white Gaussian noise. (b) Denoised Lena with 25.4 dB noise, using a soft threshold in a single basis. (c) Denoised Lena with 24.2 dB noise, using cycle spinning (frame) from [36]. (d) Denoised Lena with 23.2 dB noise, using differential cycle spinning (frame). The technique used here is an extension of the work in [64]. (Figure courtesy of Vivek Goyal).

$$\sigma_{tot}^2 = \sum_{i,j=1}^m |\langle \varphi_i, \varphi_j \rangle|^2 - mn^2 = FP(\{\varphi_i\}_{i=1}^m) - mn^2.$$

In the above, we recognize the frame potential given in [73]. The goal is to minimize the interferences and make them equal.

It is obvious that no interference is possible if and only if all φ_i are orthogonal, and in turn, this is possible only if $m \leq n$, or, when φ_i either form an orthogonal set or an ONB. When $m > n$, $FP - mn^2 \geq FP - m^2n$ and the result is known as Welch's Bound. The question of which expression is the actually Welch's Bound frequently leads to confusion. In his original paper [105], Welch found the lower bound on the maximum value of the cross-correlation of complex sequences, given in (4). In 1992, Massey and Mittelholzer [82] rephrased it in terms of the bound on the maximum user interference as given in (7). The sequences all have the same norm and

$$\sum_{i,j=1}^m |\langle \varphi_i, \varphi_j \rangle|^2 \geq m^2n, \quad (7)$$

with equality if and only if the $m \times n$ matrix Φ^* whose rows are φ_i^* has orthogonal columns of norm \sqrt{n} . If we normalize every vector to be unit norm, we can immediately translate the above into frame parlance (see Theorem 2, [73]): (a) Welch's Bound is equivalent to the frame potential inequality. (b) Frame potential is minimized at tight frames. (c) $m \times n$ matrix is the analysis operator Φ^* . (d) Columns of the analysis operator of a tight frame are orthogonal (consequence of the Naimark Theorem [73]).

This work was followed by many others, among those, by Viswanath and Anantharam's [102] discovery of the

FRAMES HAVE BEEN USED IN CDMA SYSTEMS.

Fundamental Inequality (see [73]) during their investigation of the capacity region in synchronous CDMA systems. The authors showed that the design of the optimal signature matrix S depends upon the powers $\{p_i = \|\varphi_i\|_{i=1}^m\}$ of the individual users. In particular, they divided the users into two classes: those that are oversized and those that are not. While the oversized users are assigned orthogonal channels for their personal use, the remaining users have their signature vectors designed so as to be Welch Bound Equality (WBE) sequences, namely, sequences which achieve the lower bound for the frame

potential, and are thus tight frames.

When no user is oversized, that is, when the Fundamental Inequality is satisfied, their problem reduces to finding a tight frame for \mathbb{H} with norms $\{\sqrt{p_i}\}_{i=1}^m$. The authors gave one solution to the problem using an explicit construction; characterization of all solutions to this problem using a physical interpretation of frame theory was given in [73].

The equivalence between UNTFs and Welch Bound sequences was shown in [96]. Waldron formalized that equivalence for general tight frames in [103], and consequently, tight frames have been referred in some works as Welch Bound sequences [98].

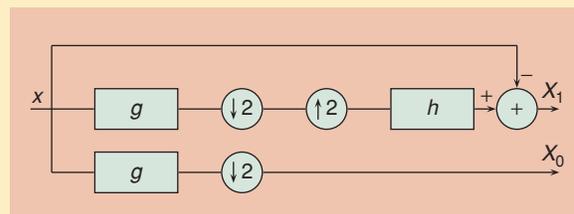
MULTIANTENNA CODE DESIGN

An important application of ENPTFs is in multiple-antenna code design [59]. Much theoretical work has been done to show that communication systems which employ multiple antennas can have very high channel capacities [52], [97]. These methods rely on the assumption that the receiver knows the complex-valued Rayleigh fading coefficients. To remove this assumption, in [61],

PYRAMID CODING

Pyramid coding was introduced in 1983 by Burt and Adelson [23]. Although redundant, the pyramid coding scheme was developed for compression of images and was recognized in the late 1980s as one of the precursors of wavelet octave-band decompositions. The scheme works as follows: First, a coarse approximation is derived (an example of how this could be done is in Figure 7). While in Figure 7 the intensity of the coarse approximation X_0 is obtained by linear filtering and downsampling, this need not be so; in fact, one of the powerful features of the original scheme is that any operator can be used, not necessarily linear. Then, from this coarse version, the original is predicted (in the figure, this is done by upsampling and filtering) followed by calculating the prediction error X_1 . If the prediction is good (which will be the case for most natural images which have a lowpass characteristic), the error will have a small variance and can thus be well compressed. The process can be iterated on the coarse version. In the absence of quantization of X_1 , the original is obtained by simply adding back the prediction at the synthesis side.

The pyramid coding scheme is fairly intuitive, thus its success. There are several advantages to pyramid coding: The quantization error depends only on the last quantizer in the iterated scheme. As we mentioned above, nonlinear operators can be used, opening the door to the whole host of possibilities (edge detectors, ...) The redundancy in 2D is only 1.33, far less than the à trous construction, for example. Thanks to the above, pyramid coding has been recently used together with directional coding to form the basis for nonseparable MD frames called contourlets (see "Multidimensional Frames").



[FIG7] The analysis part of the pyramid FB [23] with orthonormal filters g and h , corresponding to a tight frame.

new classes of unitary space-time signals are proposed. If we have n transmitting antennas and we transmit in blocks of m time samples (over which the fading coefficients are approximately constant), then a constellation of K unitary space-time signals is a (weighted by \sqrt{m}) collection of $n \times m$ complex matrices $\{\Phi_k\}$ for which $\Phi_k \Phi_k^* = I$, a PTF in other words. The i th row of any Φ_k contains the signal transmitted on antenna i as a function of time. The only structure required in general is the time-orthogonality of the signals.

Originally it was believed that designing such constellations was a too cumbersome and difficult optimization problem for practice. However, in [61], it was shown that constellations arising in a systematic fashion can be found with relatively little effort. Systematic here means that we need to design high-rate space-time constellations with low encoding and decoding complexity. Full transmitter diversity (that is, where the constellation is a set of unitary matrices whose differences have nonzero determinant) is a desirable property for good performance. In a tour-de-force, in [59], the authors used fixed-point-free groups and their representations to design high-rate constellations with full diversity. Moreover, they classified all full-diversity constellations that form a group, for all rates and numbers of transmitting antennas.

OTHER APPLICATIONS

We now just briefly touch upon a host of other applications from standard to fairly esoteric ones such as quantum teleportation.

Although unintuitive, frames were used for compression in the 1980s (unintuitive since frames are redundant and the whole purpose of compression is to remove redundancy). Burt and Adelson proposed pyramid coding of images [23] which used redundant linear transforms and was quite successful for a while (see “Pyramid Coding”).

If one considers the segmentation problem as classification into object and background, the work of [76], [99] then uses frames for segmentation. In a more recent work, de Rivaz and Kingsbury use the the complex wavelet transform (see “The Dual-Tree Complex Wavelet Transform”) to formulate the energy terms for the level-set based active contour segmentation approach [39]. They use a limited redundancy transform with a fast implementation. Both Laine and Unser used frames to decompose textures in order to characterize them across scales [76], [99]. In [28] the authors use frames for image interpolation and resolution enhancement. In [29], the authors use frames to significantly improve the classification accuracy of protein subcellular location images to close to 96%, as well as the high-throughput tagging of *Drosophila* embryo developmental stages [68].

Regularized inversion problems such as deblurring in noise can also greatly benefit from the ability of redundant frames to provide signal models that allow Bayesian regularization constraints to be applied efficiently to complicated signals such as images, as illustrated in [40].

Another application of frames has been in signal reconstruction from nonuniform samples (see [2], [8], and [49] and references therein). Benedetto and Pfander used redundant wavelet transforms (frames) to predict epileptic seizures [9], [10]. Kingsbury used his complex wavelet transform for restoration and enhancement [69], motion estimation [80] as well as building feature, texture and object detectors for images [3], [48], [72]. Balan, Casazza and Edidin used frames for signal reconstruction without noisy phase within speech recognition problems [5]. Many connections have been made between frames and coding theory [60], [87].

Recently, certain quantum measurement results have been recast in terms of frames [47], [93]. They have applications in quantum computing and have to do with positive operator valued measures (POVMs). The SIC-POVMs as well as mutually unbiased bases were discussed in “Grassmanian Packings and Equiangular Frames.” Who knows, maybe Star Trek comes to life, and frames play a role in quantum teleportation [21]!

CONCLUSIONS

As we conclude Part II of our introduction into frames and applications, we necessarily repeat what we said at the end of Part I [73]: Frames are here to stay; as wavelet bases before them, they are becoming a standard tool in the signal processing toolbox, spurred by a host of recent applications requiring some level of redundancy. We hope this article will be of help when deciding whether frames are the right tool for your application.

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