

# Perfect Reconstruction Filter Banks with Rational Sampling Factors

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**Abstract**—This paper solves an open problem, namely, how to construct perfect reconstruction filter banks with rational sampling factors. Such filter banks have  $N$  branches, each one having a sampling factor of  $p_i/q_i$  and their sum equals to one. In this way, the well-known theory of filter banks with uniform band splitting is extended to allow for nonuniform divisions of the spectrum. This can be very useful in the analysis of speech and music. The theory relies on two transforms, 1 and 2. While Transform 1, when applied, leads to uniform filter banks having polyphase components as individual filters, Transform 2 results in a uniform filter bank containing shifted versions of same filters. This, in turn, introduces dependencies in design, and is left for future work. As an illustration, several design examples for the  $(2/3, 1/3)$  case are given. Filter banks are then classified according to the possible ways in which they can be built. It is also shown that some cases cannot be solved even with ideal filters (with real coefficients).

## I. INTRODUCTION

THE most studied case of filter banks is the one with integer sampling factors. However, if one wants to analyze the signal into unequal subbands (for example, in acoustics [1]), rational sampling factors have to be allowed (see Fig. 2(a)). Then, each channel would have a sampling factor  $p_i/q_i$  and their sum equals to one (so as to preserve the sampling density). For some cases, it was known how to solve this problem in practice, since one could divide the spectrum into  $Q = \text{lcm}(q_i)$  parts and then resynthesize the appropriate subspectra. However, because this approach is indirect, it is suboptimal in terms of computational complexity and filter quality, and not all cases can be solved. Previous work in this area was aimed at aliasing cancellation [2], solutions that are built through tree splitting [3] and numerical solutions [4].

In this paper we present a direct method for designing perfect reconstruction filter banks with rational sampling factors. It relies on two transforms, 1 and 2. While Transform 1, when applied, leads to uniform filter banks having polyphase components as individual filters, Transform 2

results in a uniform filter bank containing shifted versions of the same filters. This, in turn, introduces dependencies in design, and is left for future work. Note that this is a continuation of previous work [5]–[7].

The outline of the paper is as follows: Section II briefly reviews necessary facts from the theory of perfect reconstruction filter banks. Section III presents an extensive discussion on realizability of various filter banks with rational sampling factors, showing possible problems. Section IV develops the direct design method and the tools that enable it. In Section V, filter banks are classified according to the way they can be built. Section VI gives several filter designs and compares them to the indirect ones.

In what follows, all filters will be assumed to have real coefficients, unless stated otherwise.

## II. A GLIMPSE AT PERFECT RECONSTRUCTION FIR FILTER BANKS

Here, we briefly recall some of the concepts from the theory of perfect reconstruction filter banks that are going to be used in the remainder of this paper. For a more extensive treatment of the subject, refer to [3], [8]–[10].

An analysis filter bank is a signal processing device that splits the input signal into  $M$  channel signals by means of filtering and downsampling by  $N$  (where  $N \leq M$ ). In what follows, we will assume that  $N = M$ , i.e., the filter bank is critically sampled. The synthesis filter bank performs the inverse task (see Fig. 1(a)). Due to the fact that the downsampling is a periodically shift-variant operation with period  $N$  (that is, if the input  $x(n)$  produces output  $y(n)$ , then the input  $x(n - n_0)$  will produce output  $y(n - n_1)$  only if  $n_0 = Nn_1$ ), the whole system becomes periodically shift variant. A way to make the analysis of such a system easier, is to decompose both signals and filters into so-called polyphase components. For a filter, each polyphase component would then represent one of  $N$  impulse responses (at times  $0, 1, \dots, N - 1$ ). Thus, a filter can be expressed as

$$H_j(z) = \sum_{i=0}^{N-1} z^{-i} H_{ji}(z^N) \quad (1)$$

where  $H_{ji}(z)$  is the  $i$ th polyphase component of the filter  $H_j(z)$ , and is given by

$$H_{ji}(z) = \sum_n h_j(nN + i) \cdot z^{-n}. \quad (2)$$

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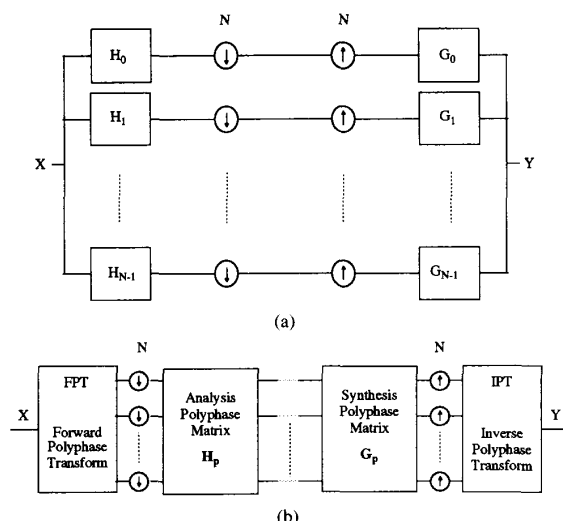


Fig. 1. (a) Analysis/synthesis filter banks. (b) Filter bank in the polyphase domain.

It turns out that the output of the system can be conveniently expressed in terms of analysis and synthesis polyphase matrices (that is, matrices containing polyphase components of analysis and synthesis filters), as well as forward and inverse polyphase transforms. Forward polyphase transform inputs the signal and outputs its  $N$  polyphase components by means of shifting and downsampling by  $N$ . The inverse polyphase transform performs the inverse task, that is, the forward and the inverse polyphase transforms are inverses of each other. Perfect reconstruction is equivalent to forcing the synthesis polyphase matrix to be the inverse of the analysis one. A filter bank expressed in the polyphase domain is given in Fig. 1(b). One of the easiest ways to achieve perfect reconstruction (i.e., to obtain the output as a perfect replica of the input), is to construct a paraunitary analysis matrix (or orthogonal, lossless). In other words, the analysis polyphase matrix has to satisfy the following [11]:

$$H_p^t(z^{-1}) \cdot H_p(z) = I. \quad (3)$$

Then the synthesis polyphase matrix can be chosen as  $z^{-k} H_p^t(z^{-1})$ , which in turn yields filters that are the same as the analysis filters (within shift reversal).

Let us also point out some facts on multirate filtering that are going to be used later.

1) Upsampling by  $p$  and downsampling by  $q$  can be interchanged if and only if  $p$  and  $q$  are relatively prime [9].

2) The output after filtering by  $H(z)$  and downsampling by  $N$  can be written as [8]

$$Y(z) = \frac{1}{N} \sum_{k=0}^{N-1} H(W_N^k z^{1/N}) \cdot X(W_N^k z^{1/N}) \quad (4)$$

where  $W_N$  denotes the  $N$ th root of unity, i.e.,  $W_N = e^{-j(2\pi/N)}$ .

3) A pair of useful identities known under the name of "noble identities" [9], gives conditions under which shift-invariant filters can be passed across up- and downsamplers. They state that any filter in the downsampled domain can be represented in the upsampled domain by simply upsampling its impulse response. Very similarly, a filter with  $z$ -transform  $H(z)$  placed in front of upsampling by  $N$  can be moved past the upsampler and represented as  $H(z^N)$  [9].

### III. FILTER BANKS WITH RATIONAL SAMPLING FACTORS

In this section, filter banks with rational sampling factors are investigated, and some intuition for possible solutions and problems is developed. In order to do so, we will often resort to ideal filters (with real coefficients), that is, filters with box function magnitude responses.

The goal is to realize a filter bank with rational sampling factors as shown in Fig. 2(a), performing a spectral analysis as depicted in Fig. 2(b). First, a fact on the position of an ideal real  $q$ th band filter in the spectrum, necessary for further analysis, is presented. This is followed by the discussion of an indirect method, which always leads to perfect reconstruction, but may produce "shuffling" of frequencies within subbands (that is, some lower frequencies may appear above higher ones). The direct method is investigated, where sometimes, there are no solutions even using ideal filters (assuming contiguous passbands and real coefficients). This leads to the question of equivalence between direct and indirect methods. Tree structures are also considered, being the simplest representatives of unequal bandwidth filter banks.

We shall use the notation  $[p_0/q_0, p_1/q_1, \dots, p_{N-1}/q_{N-1}]$  to denote a filter bank where the  $i$ th channel (numbered with increasing frequency) has rate  $p_i/q_i$ , and contains input frequencies ranging over

$$\left[ \left( \sum_{k=0}^{i-1} \frac{p_k}{q_k} \right) \pi, \left( \sum_{k=0}^i \frac{p_k}{q_k} \right) \pi \right] \quad (5)$$

where the sum is defined to be 0 if the upper bound is negative. We will also assume that the filter bank is critically sampled, that is

$$\sum_{i=0}^{N-1} \frac{p_i}{q_i} = 1. \quad (6)$$

Note also that "mirroring" will describe the process in which the spectrum is the reversed version of the true spectrum, that is,  $|X_{\text{mirrored}}(\omega)| = |X(\omega + \pi)|$  (assuming real signals). "Shuffling," however, will denote the process in which a part of the signal's spectrum has been translated to another part in the spectrum. For example, assume that the signal occupies the part of the spectrum from  $[0, 2\pi/3]$ , and after processing, the part from  $[0, \pi/3]$  is moved to  $[2\pi/3, \pi]$ . Thus, the spectrum that the signal occupies after processing is from  $[\pi/3, \pi]$ . However, the ordering of the frequencies inside the signal spectrum has been reshuffled.

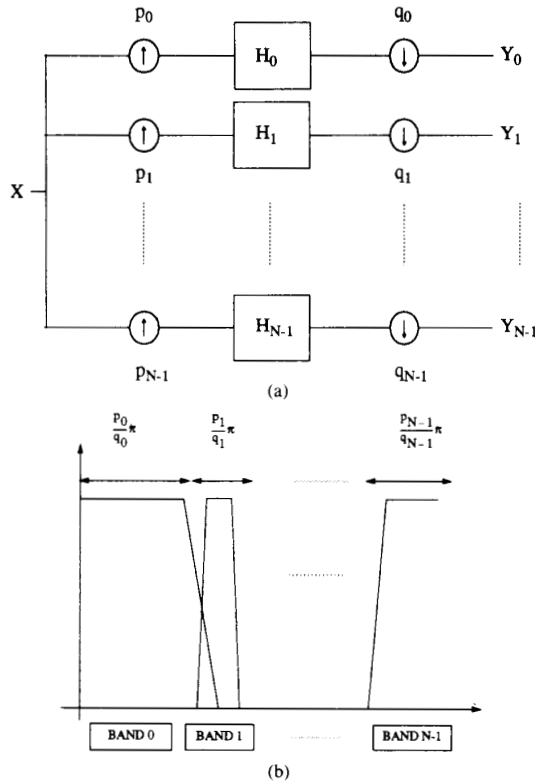


Fig. 2. Filter bank with rational sampling factors: (a) a block diagram and (b) the desired spectrum splitting.

#### A. Position of the Ideal $q$ th Band Filter with Real Coefficients in the Spectrum

Consider a branch consisting of an ideal bandpass filter  $H$  (with real coefficients), downsampler by  $q$ , unsampler by  $q$ , and the same bandpass filter  $H$ . By the sampling theorem,  $H$  has to be a  $q$ th band filter in order to avoid aliasing, and moreover:

**Proposition 3.1:** A  $q$ th band filter  $H$  with real coefficients, meant to avoid aliasing in the system as above, has to be situated at

$$H: \left[ \frac{s}{q} \pi, \frac{s+1}{q} \pi \right] \quad (7)$$

for some  $s \in \{0, \dots, q-1\}$  (symmetrically for negative frequencies).

The proof is given in Appendix B.

#### B. Indirect Method

An easy way to achieve the desired factors of Fig. 2(a) is as follows; Call  $Q$  the least common multiple of all the downsampling factors, and analyze the input into

$$Q = \text{lcm}(q_0, q_1, \dots, q_{N-1}) \quad (8)$$

subbands. Define

$$p'_i = \frac{p_i \cdot Q}{q_i} \quad (9)$$

Then, to obtain a perfect reconstruction filter bank, one can combine an analysis filter bank having  $Q$  filters with  $N$  synthesis filter banks, each with  $p'_i$  filters ( $i = 0, \dots, N-1$ ). Such a scheme is shown in Fig. 3. Obviously, if analysis and synthesis banks are perfect reconstruction, the overall system will be perfect reconstruction as well. This purely algebraic result does not guarantee, however, that the spectral analysis performed by the filter bank is meaningful in the sense that a subband represents a piece of the input spectrum (as is, for example, the case for uniform filter banks).

To give a flair of possible problems, consider first an indirect design of a filter bank with sampling factors  $(2/3, 1/3)$  (that is, the low-pass filter of width  $2\pi/3$  and the high-pass filter of width  $\pi/3$ ), as shown in Fig. 4(a). It can be verified that the low-pass channel is a "faithful" representation of the frequencies from 0 to  $2\pi/3$  in the input signal, when the synthesis filter bank uses a low-pass filter for channel 0 and a high-pass for channel 1 in the synthesis bank (see Fig. 4(c)). By "faithful" we mean that the range of input frequencies is conserved in the correct order.

Note that if the synthesis would instead use high-pass and low-pass filters for channels 0 and 1 respectively, the system would still be perfect reconstruction, but now, the spectrum in the rate  $2\pi/3$  subband would be the mirror image of the true spectrum of the signal (that is, "high" frequencies would appear before "low" ones).

Consider now a filter bank with sampling factors  $(1/3, 2/3)$ , that is, the high-pass filter now uses recombination (see Fig. 5(a)). From the spectra available at the input of the synthesis bank, it can be seen that the original spectral relations cannot be preserved. With a low-pass/high-pass pair, one gets a shuffled and mirrored version of the spectral band of interest, while with a high-pass/low-pass pair, low and high bands are interchanged. It is important to note that while we still have perfect reconstruction, there is no simple "bandpass" interpretation anymore due to the spectral shuffling and mirroring.

To formalize the previous discussion, we will show now under which condition the indirect method works. To start, we first state some assumptions. In what follows, the synthesis banks used for recombination will contain bandpass filters with central frequencies in increasing order, that is, the situations as the one described in Fig. 4(c) with  $(G_0, G_1)$  a high-pass/low-pass pair, are not considered. Then, we will state the result for a single subband, that is  $P = p'_i$  branches from the analysis bank, followed by size  $p'_i$  synthesis bank (and we assume  $P > 1$ ). For the whole system to work, the conditions that follow have to be satisfied for  $i = 0, \dots, N-1$ . Note that here,  $P$  and  $Q$  are not necessarily prime (as can be seen from (9)).

Therefore, suppose we want to extract the following part of the input signal:

$$X_R: \left[ \frac{k\pi}{Q} \pm 2n\pi, \frac{(k+P)\pi}{Q} \pm 2n\pi \right] \quad (10)$$

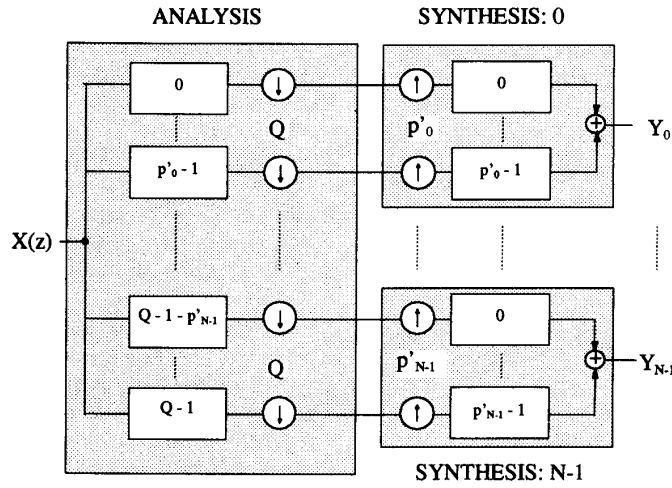


Fig. 3. Nonuniform filter bank designed indirectly.

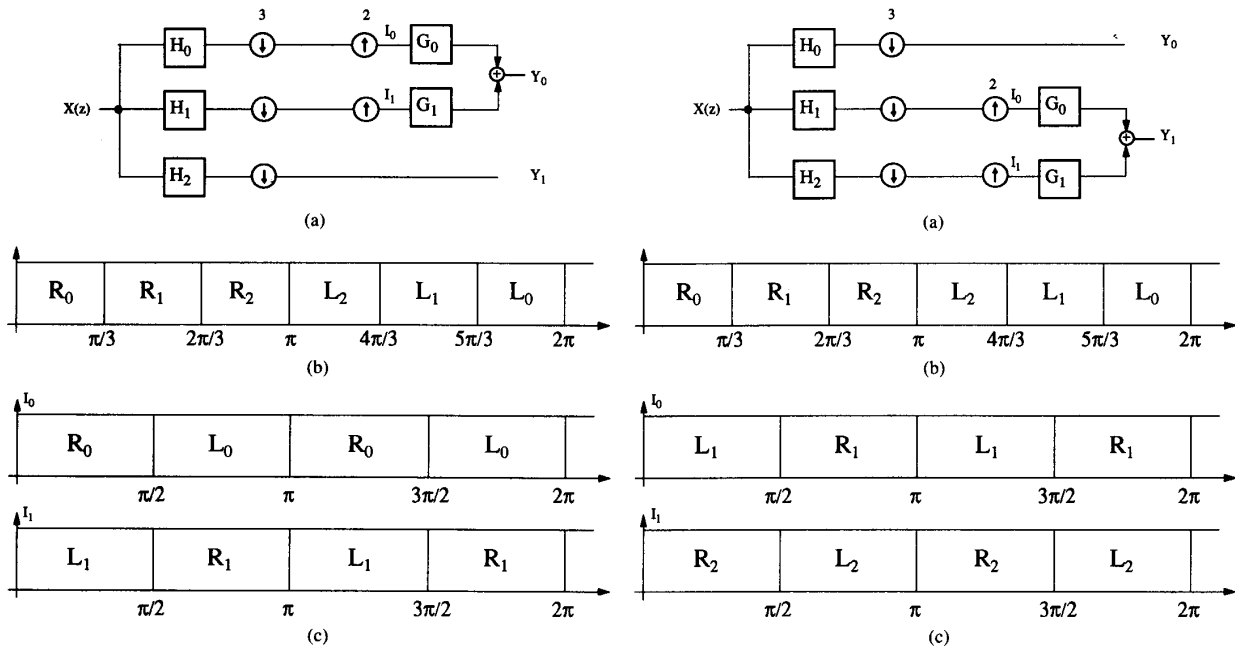


Fig. 4. Indirect method with ideal filters for (2/3, 1/3) splitting. (a) Three-band analysis followed by two-band synthesis of the low-pass filter. (b) Original spectrum indicating the pieces retained by the analysis filter bank. (c) Spectra before entering  $G_0$  and  $G_1$ . If  $(G_0, G_1)$  is a low-pass/high-pass pair, the spectrum of  $Y_0$  will be a correct representation of the input spectrum. If it is a high-pass/low-pass pair instead, the spectrum will be incorrect.  $R$  and  $L$  denote the right- and left-hand sides of the magnitude responses, respectively.

by means of  $P$  analyzing ideal bandpass filters and the same number of synthesizing ones ( $X_R$  denotes the right-hand side of the spectrum). After downsampling by  $Q$  followed by upsampling by  $P$ , in the first branch, aliased versions will appear at

$$X_R: \left[ \frac{k \pm 2nQ - 2i}{P} \pi, \frac{(k+1) \pm 2nQ - 2i}{P} \pi \right]. \quad (11)$$

Fig. 5. Indirect method with ideal filters for (1/3, 2/3) splitting. (a) Three-band analysis followed by two-band synthesis of the high-pass filter. (b) Original spectrum indicating the pieces retained by the analysis filter bank. (c) Spectra before entering  $G_0$  and  $G_1$ . Neither a low-pass/high-pass nor a high-pass/low-pass pair will preserve the original spectral relations.

To avoid shuffling of frequencies, one of the  $X_R$ 's has to descend to  $[0, \pi/P]$ , since otherwise, its mirrored version will appear in that region, followed by the ones from the channels 1 through  $P-1$ . This means that

$$\frac{k \pm 2nQ - 2i}{P} \pi = 0 \quad (12)$$

or

$$k = 2i \mp 2nQ \quad (13)$$

and thus:

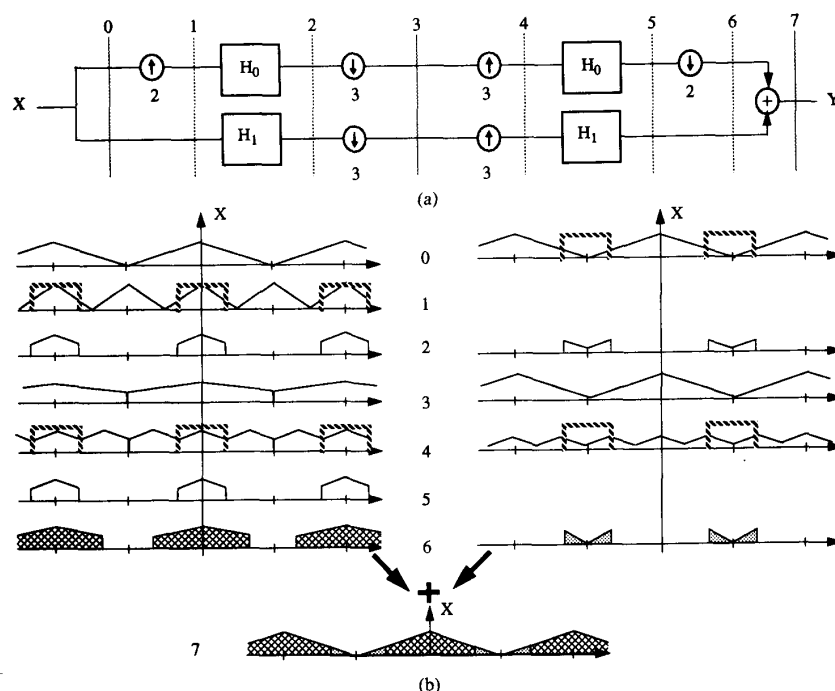


Fig. 6. Perfect reconstruction obtained using ideal filters. (a) Filter bank and (b) spectral analysis (shaded lines represent magnitude responses of ideal filters).

**Proposition 3.2:** Not to have shuffling of frequencies in the indirect method,  $k$  (as defined above), has to be even (assuming  $P > 1$ ).

This result immediately disqualifies the case with sampling factors  $(1/3, 2/3)$  considered earlier, since the second subband should cover  $[\pi/3, \pi]$ , i.e.,  $k = 1$ . However, the case  $(1/6, 1/6, 4/6 = 2/3)$ , that has the same high-pass, is possible.

Note here that this method, when it works, i.e., when it does not produce frequency shuffling, is suboptimal in terms of computational complexity and filter quality. It will be shown later that in some cases, one can transform this system into one as in Fig. 2(a), thus being able to control the characteristics of the equivalent nonuniform bandpass filters.

### C. Direct Method

Consider a direct implementation of the  $(2/3, 1/3)$  filter bank as shown in Fig. 6. Using ideal filters (with real coefficients) one gets perfect reconstruction as can be seen from the spectra in Fig. 6. However, if one tries to implement a  $(1/3, 2/3)$  filter bank (with a low-pass of width  $\pi/3$  and a high-pass of width  $2\pi/3$ , see Fig. 7(a)), then aliasing cannot be avoided and perfect reconstruction is impossible. The reason is that the spectrum of interest is situated, after upsampling by 2, at  $[\pi/6, \pi/2]$ , which by Proposition 3.1 cannot be subsampled by 3 without aliasing (see Fig. 7(b)). Using a more complicated filter for the high-pass channel instead, (as shown in Fig. 7(c)) al-

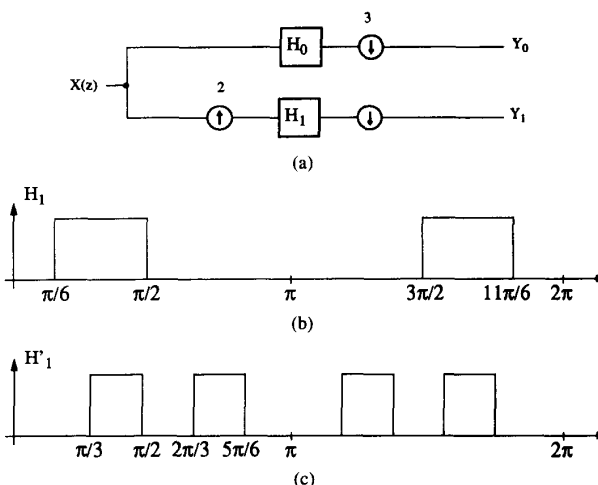


Fig. 7. Direct method for the  $(1/3, 2/3)$  splitting. (a) Analysis bank. (b) Band-pass filter for spectrum of interest leading to aliasing when downsampled by 3. (c) Another filter for spectrum of interest which does not lead to aliasing, but leads to shuffling of frequencies and does not have contiguous spectrum.

lows one to downsample by 3 without aliasing, thus permitting perfect reconstruction, but still producing frequency shuffling in  $Y_1$ . Therefore, in what follows, we will assume only filters with contiguous passbands, hence excluding the one from Fig. 7(c). The previous discussion showed that even in the simplest, two-channel case, a number of problems can arise. Thus, the question we want

to ask now is, given a branch with upsampling by  $p$  followed by an ideal bandpass filter  $H$  (with contiguous spectrum and real coefficients) and downsampling by  $q$ , with  $H$  satisfying Proposition 3.1, which part of the input signal's spectrum can we extract? Note that  $p$  and  $q$  are assumed to be coprime.

To get an answer, note first that by Proposition 3.1, the available bandpass filters reside from

$$H: \left[ \frac{s}{q} \pi, \frac{s+1}{q} \pi \right], \quad s = 0, \dots, q-1. \quad (14)$$

Call a "half-image" of the input signal either its part situated at positive frequencies or the part at negative frequencies. To get a valid part of the input signal's spectrum, they have to fall within one of the half-images of the input signal after upsampling, that is, if a half-image is from

$$\left[ \frac{l}{p} \pi, \frac{l+1}{p} \pi \right], \quad l = 0, \dots, p-1 \quad (15)$$

then the following has to be satisfied:

$$\frac{l}{p} \pi \leq \frac{s}{q} \pi, \quad \frac{s+1}{q} \pi \leq \frac{l+1}{p} \pi \quad (16)$$

or, in other words

$$(s+1) \frac{p}{q} - 1 \leq l \leq s \frac{p}{q}. \quad (17)$$

This leads to the following proposition:

**Proposition 3.3:** Assuming available bandpass filters are given in (14), if there is  $l \in \{0, p-1\}$ , such that (17) is satisfied, then the following part of the signal is extracted:

$$X_s: \left[ \left( s \frac{p}{q} - l \right) \pi, \left( (s+1) \frac{p}{q} - l \right) \pi \right] \quad (18)$$

for even  $l$ , or

$$X_s: \left[ \left( l+1 - (s+1) \frac{p}{q} \right) \pi, \left( l+1 - s \frac{p}{q} \right) \pi \right] \quad (19)$$

for  $l$  odd.

The question can also be posed in the other direction, i.e., is it possible to extract a particular part of the signal, and if it is, which is the bandpass filter to use.

**Proposition 3.4:** It is possible to extract the following part of the input signal's spectrum

$$X_o: \left[ \frac{o}{q} \pi, \frac{o+p}{q} \pi \right] \quad (20)$$

if there exist  $l, s$  such that one of the following is satisfied:

$$o = sp - lq, \quad l = 2t \quad (21)$$

$$o - q + p = lq - sp, \quad l = 2t + 1 \quad (22)$$

where  $l = 0, \dots, p-1, s = 0, \dots, q-1$ , and  $o = 0, \dots, q-p$ . If there is a solution, the filter to be used

is

$$H: \left[ \frac{s}{q} \pi, \frac{s+1}{q} \pi \right]. \quad (23)$$

As an example consider Fig. 7. We would like to extract  $[\pi/3, \pi]$  from the signal, i.e.,  $o = 1$ , and  $l = 0, 1$ . It is easy to see that in this case neither (21) nor (22) can be satisfied, and thus we are not able to extract the desired part of the input signal's spectrum.

As for the indirect method, the direct method has to be checked for each branch.

#### D. Indirect versus Direct Method

The intent in this section is to make a connection between the indirect and the direct methods. In order to do this, we will use the partial solution developed in [5].

Start with the indirect method as described in Section III-B and assume that all the  $p_i$ 's are coprime with  $Q$  (this will not be the case in general). But  $Q$  can be written as

$$Q = \text{lcm}(q_0, q_1, \dots, q_{N-1}) = r_i q_i,$$

$$i = 0, \dots, N-1$$

and thus

$$p'_i = \frac{p_i \cdot Q}{q_i} = r_i p_i.$$

Now if one wants  $p'_i$  and  $Q$  to be coprime for  $i = 0, \dots, N-1$ ,  $r_i$  has to equal to 1,  $i = 0, \dots, N-1$ , and this in turn implies

$$q_0 = \dots = q_{N-1} = q \quad (24)$$

and

$$p'_i = p_i, \quad i = 0, \dots, N-1. \quad (25)$$

Conditions (24) and (25) are necessary for the transformations that follow to work. On the other hand, for the indirect method to work, one can use Proposition 3.2 to conclude that  $p_i$ 's have to satisfy the following conditions (C1):

- 1) If  $q$  is even, then  $p_{2i}$  is odd and  $p_{2i+1} = 1$ .
- 2) If  $q$  is odd, then if  $p_{2i}$  is odd,  $p_{2i+1} = 1$ , or if  $p_{2i}$  is even,  $p_{2i+1}$  is even as well.

In Fig. 3, consider the block  $i$  which has the rational sampling factor  $p_i/q_i = p_i/q$ . Since  $p_i$  and  $q$  are coprime, upsampling and downsampling can be interchanged. Then, as explained in Section II, upsampling can be moved across the analysis filters (raising  $z$  to the  $p_i$ th power), and the downsampling across the synthesis filters (raising  $z$  to the  $q$ th power). This results in  $p_i$  filters in parallel, each preceded by upsampling by  $p_i$  and followed by downsampling by  $q$ . This is equivalent to a single filter with the transfer function

$$H_{e_i}(z) = \sum_{j=0}^{p_i-1} H_j(z^{p_i}) \cdot G_j(z^q) \quad (26)$$

where  $j' = \sum_{k=0}^{p_i-1} p_k + j$  and the filters in the analysis and synthesis banks are denoted by  $H$  and  $G$ , respectively. As

an example, for the  $(2/3, 1/3)$  case, the equivalent low-pass filter would be given by

$$H_{e0}(z) = H_0(z^2)G_0(z^3) + H_1(z^2)G_1(z^3). \quad (27)$$

This is obviously a powerful tool since one can design the filter bank having the complete control over the filters' magnitude responses. Unfortunately, this method does not cover all possible cases since all the  $p_i$ 's will not in general be coprime with  $Q$ . Another point is that both an analysis and a synthesis bank are being designed for a system that has to perform only analysis.

Now we are ready to show the following.

**Proposition 3.5:** If (24) and (25) are satisfied, and if the indirect method works, the direct one works as well.

The proof is given in Appendix B. Unfortunately, the other direction is not true, i.e., the direct method working does not imply that the indirect one is working as well. A simple counterexample is the case with sampling factors  $(3/7, 3/7, 1/7)$ . It can be checked that the direct method will work, while the indirect one will not (the middle channel starts at  $k = 3$  violating Proposition 3.2).

Note that the discussion in this section was based on the partial solution, that is, on the assumption that all downsampling factors are the same and all  $p_i$ 's are coprime with  $q$ . When this is not the case, the indirect method can work, e.g.,  $(1/6, 1/6, 4/6, = 2/3)$ , while the direct one will not.

#### E. Miscellaneous Remarks

1) **Tree Structures:** If we say that a filter bank has a tree structure, it will mean that its sampling factors could be obtained by cascading uniform filter banks. A necessary condition for a tree structure is

$$p_0 = \dots = p_{N-1} = 1. \quad (28)$$

Beside testing whether all upsampling factors are 1, one has to test further whether the set of downsampling factors  $(q_0, \dots, q_{N-1})$  represents a tree. To that end, Fig. 8 shows the procedure to determine that fact, that is, the routine that is recursively used for testing first the main tree and then branching into subtrees. Note that ordering is important, that is, for  $(2, 4, 4)$  the algorithm will give a positive answer, while for  $(4, 2, 4)$  it will not.

Just by virtue of the fact that in this case the same filter bank could be built by cascading uniform filter banks (although with possibly different filters), these structures are always realizable (since there is no difference when ideal filters are used). However, a slight problem might arise, one that we will call "permutation of the bands." Consider the case with sampling factors  $(1/2, 1/4, 1/4)$ . It is easy to check that the low-pass from  $[0, \pi/2]$ , band-pass from  $[\pi/2, 3\pi/4]$  and high-pass from  $[3\pi/4, \pi]$ , since positioned properly in the spectrum, will not produce aliasing. If the same system were built using two cascaded filter banks, both of them having ideal low-pass and high-pass filters, the first channel would still be the same, the second one (high-pass followed by a low-pass)

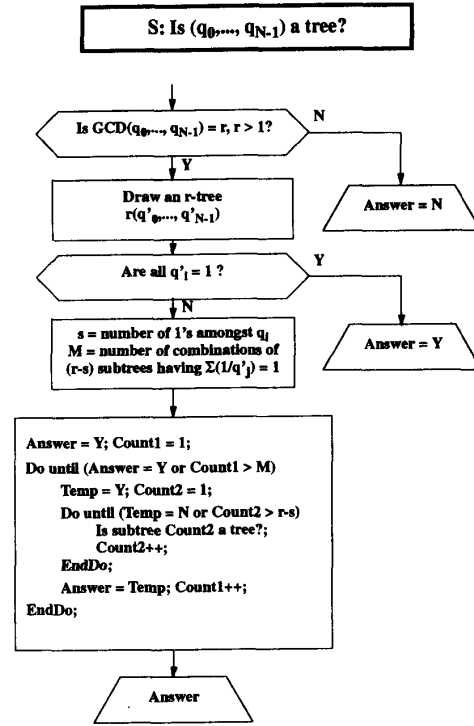


Fig. 8. The routine that determines whether a set of downsampling factors represents a tree. The filter banks is assumed to be critically sampled.

will be from  $[3\pi/4, \pi]$ , and the last one would reside from  $[\pi/2, 3\pi/4]$ . As can be seen, the second and the third channel have changed place. This is only an artificial problem though, since resolving it consists only in re-naming, or permuting, the bands.

Note also that in these cases, there is a subtle question in design, that is, lower complexity filters (obtained when filter banks are cascaded) versus more general ones (obtained when the design is direct).

2) **Compatible Sets [2]:** In [2], the authors have presented the concept of "compatible sets," that we summarize here for completeness.

**Definition 3.1 [2]:** Let  $S = \{q_0, \dots, q_{N-1}\}$  be an ordered set of positive integers,  $q_0 \leq q_1 \leq \dots \leq q_{N-1}$ .  $S$  is compatible if it satisfies the following conditions:

$$1) \sum_{i=0}^{N-1} 1/q_i = 1.$$

$$2) \text{ For every } q_i, l_i, (l_i \leq q_i - 1), \text{ there exist } q_j, l_j, (l_j \leq q_j - 1), \text{ with } q_i \neq q_j, \text{ such that } W_{q_i}^{l_i} = W_{q_j}^{l_j}.$$

The concept of compatible sets arises when looking at the "alias cancellation" matrix in the  $z$  domain. It immediately tells that the case  $(1/2, 1/3, 1/6)$  is not realizable. The authors also state that every tree-structured filter bank will produce a compatible set, but the converse is not true. For more details, refer to [2].

3) **Existence of the Polyphase Transform:** Moving away for a moment from the ideal filters, assume that they are simple delays instead, and try to obtain the polyphase transform, that is, try to represent the input signal completely by its polyphase components with respect to var-

ious sampling factors. We will just mention two simple cases in order to hint at possible problems.

As stated earlier, the case  $(1/2, 1/3, 1/6)$  cannot be realized with ideal filters. However, permuting the factors as follows  $(1/2, 1/6, 1/3)$ , one obtains a system in which all bandpass filters conform to the requirement of Proposition 3.1, i.e., aliasing is canceled with ideal filters. At the other end of the spectrum, one would like to find the polyphase transform, which turns out to be impossible for both cases. Note that the existence of polyphase transform does not depend on the ordering of the sampling factors, and thus in the above, although one case can and the other cannot be realized with ideal filters, none of them possesses a polyphase transform.

Another interesting case is with sampling factors  $(1/2, 1/4, 1/4)$ , which has an ideal solution, while its permutation  $(1/4, 1/2, 1/4)$  does not. However, since we know that the first one has a polyphase transform, the second one will have it as well.

4) *Modulation*: Note that modulation is not used in the filter banks we are considering, thus down- and upsamplers are the only periodically shift-varying devices in the system. If modulation were used, however, one could solve most of the cases, by shifting (modulating) filter's spectrum to the required positions.

5) *Mirroring in Uniform Filter Banks*: Let us stress once more the fact that has been overlooked in the liter-

2) All signals are assumed to be real.

3) All filter banks are critically sampled.

4) In the indirect method, the synthesis banks used for recombination will contain bandpass filters with central frequencies in increasing order.

5) Unless otherwise stated, the upsampling and downsampling factors in a single branch  $(p_i, q_i)$ , are coprime.

6) Modulation is not used to achieve desired solutions.

#### IV. A DIRECT DESIGN METHOD

##### A. Towards a General Solution

In what follows, we will be dealing only with realizable cases (as explained in Section III). The ideal filters discussed previously, although conceptually natural, are clearly not practical. So the question now is whether a system with rational sampling factors can be realized with finite length filters. The answer has been known to be positive in the case with sampling factors  $r_i = 1/N$ ,  $i = 0, \dots, N-1$ . The purpose of this section is to show the realizability in two specific cases that will lead to the direct design method presented in Section IV-B.

Consider first the case with sampling factors

$$r_0 = \frac{2}{3}, \quad r_1 = \frac{1}{3} \quad (29)$$

and note that the output of the upper branch at point 3 in Fig. 6,  $y_0(n)$ , can be written as follows:

$$\begin{pmatrix} \vdots \\ y_0(n) \\ y_0(n+1) \\ y_0(n+2) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \cdots & h_0(L-1) & h_0(L-3) \\ \cdots & 0 & 0 \\ \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots \\ h_0(L-5) & h_0(L-7) & \cdots \\ h_0(L-2) & h_0(L-4) & \cdots \\ 0 & h_0(L-1) & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ x(n) \\ x(n+1) \\ x(n+2) \\ \vdots \end{pmatrix}. \quad (30)$$

ature dealing with uniform filter banks, that is, that for odd indices, the spectra after downsampling will be the mirrored versions of the corresponding parts of the input signal's spectrum (see the concluding remarks in Section III-A). As noted earlier, if one would need a true representation it would suffice to reverse the spectrum, or, multiply the sequence by  $(-1)^n$ .

6) *Word of Care*: The present section, although in no way complete, was meant to give a glimpse at possible problems arising if one wants to deal with filter banks having rational sampling factors. It was shown, that even in very simple cases, one has to be very careful and aware of possible problems (mirroring, reshuffling of frequencies, permutation of the bands), and that some cases cannot be solved, even with ideal filters.

The following is the list of assumptions that are going to be used throughout the rest of the paper:

1) All filters are assumed to be FIR with real coefficients and contiguous passbands.

It is worth noting here that the outputs  $y_0(2n)$  are obtained by convolving  $h_0(2n)$  with  $x(n)$ , while  $y_0(2n+1)$  is the result of the convolution of  $h_0(2n+3)$  with  $x(n)$ , suggesting that the filter  $h_0$  actually behaves as two filters in parallel  $h_{00}$  and  $h_{01}$  ( $h_{0i}$  being the  $i$ th polyphase component of  $h_0$ ). This reasoning led to Transform 1 given in Section IV-B. Note, however, that the fact that this case leads to the filter being implemented as its two polyphase components in parallel (appropriately shifted), is due to the fact that the up/downsampling factors are relatively prime (not true in general).

Thus, the whole system could be implemented as follows (see Fig. 9(a)): design a 3-channel filter bank (with downsampling by 3), where the first filter is the first polyphase component of the desired filter  $h_0$ , the second filter is the shifted second polyphase component of  $h_0$ , and the third filter is  $h_1$ . To obtain the channel signal  $y_0$ , one has to interleave the outputs of the first two branches, this process being equivalent to upsampling the two outputs,



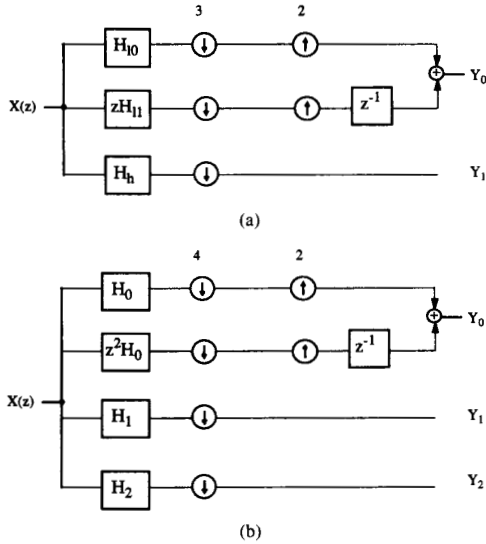


Fig. 9. Direct designs of the filter banks with sampling factors (a) (2/3, 1/3) and (b) (1/2, 1/4, 1/4).

delaying the second one and putting them together. This, in turn, is equivalent to passing the channel signals through the inverse polyphase transform (IPT) blocks. Let us now try to perform the same analysis in the case with sampling factors

$$r_0 = \frac{1}{2}, \quad r_1 = r_2 = \frac{1}{4}. \quad (31)$$

Note that this filter bank could be built using a tree. The aim here is to see whether it can be implemented directly instead, in such a way that other solutions are possible. The channel signals can now be expressed as

$$\begin{pmatrix} \vdots \\ y_0(n) \\ y_0(n+1) \\ y_1(n) \\ y_2(n) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & h_0(L-1) & h_0(L-2) & h_0(L-3) & h_0(L-4) & \cdots \\ \cdots & 0 & 0 & h_0(L-1) & h_0(L-2) & \cdots \\ \cdots & h_1(L-1) & h_1(L-2) & h_1(L-3) & h_1(L-4) & \cdots \\ \cdots & h_2(L-1) & h_2(L-2) & h_2(L-3) & h_2(L-4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ x(n) \\ x(n+1) \\ x(n+2) \\ x(n+3) \\ \vdots \end{pmatrix}. \quad (32)$$

In other words, such a case could be implemented as a 4-channel filter bank where the first two filters are  $h_0$  and its shifted version by 2, while the last two filters are  $h_1$  and  $h_2$  (see Fig. 9(b)). This type of implementation led to Transform 2, presented in Section IV-B. Note also, that this case is fundamentally different from the previous one, in that the first two filters are obtained just by shifting a single filter, the fact that introduces dependencies and problems in design.

#### B. Transforms for Implementing the Direct Solution

As shown in Section IV-A, there is a way to implement directly the filter bank with rational sampling factors (2/3,

1/3). The aim here is to extend this approach into transforming a single branch with upsampling by  $p$  and downsampling by  $q$  using a  $p$ -channel analysis bank with downsampling by  $q$  and an inverse polyphase transform of size  $p$ , assuming  $p$  and  $q$  are coprime and  $p > 1$ . As in the last section, one can see that the output of the branch is the convolution of the input signal and a particular polyphase component of the filter, and thus the filter can be expressed as  $p$  filters in parallel, each one of them being (see Fig. 10(a))

$$H'_i(z) = z^{d_i} H_i(z) \quad (33)$$

where  $d_i = \lfloor q \cdot i/p \rfloor$  ( $\lfloor x \rfloor$  denotes the largest integer not greater than  $x$ ),  $t_i = q \cdot i \bmod p$  and  $H_0, \dots, H_{p-1}$  are the polyphase components of  $H$  with respect to  $p$ . It should be noted that for  $p = 1$  there is no transform, i.e.,  $H'_0 = z^{d_0} H_0 = z^0 H_0 = H$  since the only polyphase component with respect to  $p = 1$  is the filter itself.

Note that the same transform, developed in a different context (minimization of the number of arithmetic operations per unit time), was first given in [12], and thus, for the proof that the two representations are equivalent, the reader is referred to [12].

At this point, one could apply Transform 1 to all branches in Fig. 2 (a). If  $q_0 = \dots = q_{N-1} = q$ , nothing else has to be done, since the transform will result in  $\sum_{i=0}^{N-1} p_i = q$  branches followed by downsampling by  $q$ . Thus, the problem has been reduced into finding a perfect reconstruction structure for a  $q$ -channel filter bank, with design constraints imposed on filters  $H_0, \dots, H_{N-1}$ .

If, however, not all  $q_i$ 's are the same, applying Transform 1 will, for each branch  $i$ , produce  $p_i$  branches fol-

lowed by downsampling by  $q_i$  (note that those with  $p_i = 1$  will remain the same). Thus, what one would like to do is to transform this into a system having  $Q$  branches followed by downsampling by  $Q$ . For example, the case given in Section IV-A with rational sampling factors (1/2, 1/4, 1/4), would remain untouched by Transform 1, and thus one wants to transform it further so as to obtain four branches followed by downsampling by 4, as was shown in Section IV-A. The same approach as the one used in that case leads to Transform 2.

Therefore, now we show how to express a single branch with downsampling by  $q$  using a  $p$ -channel analysis bank

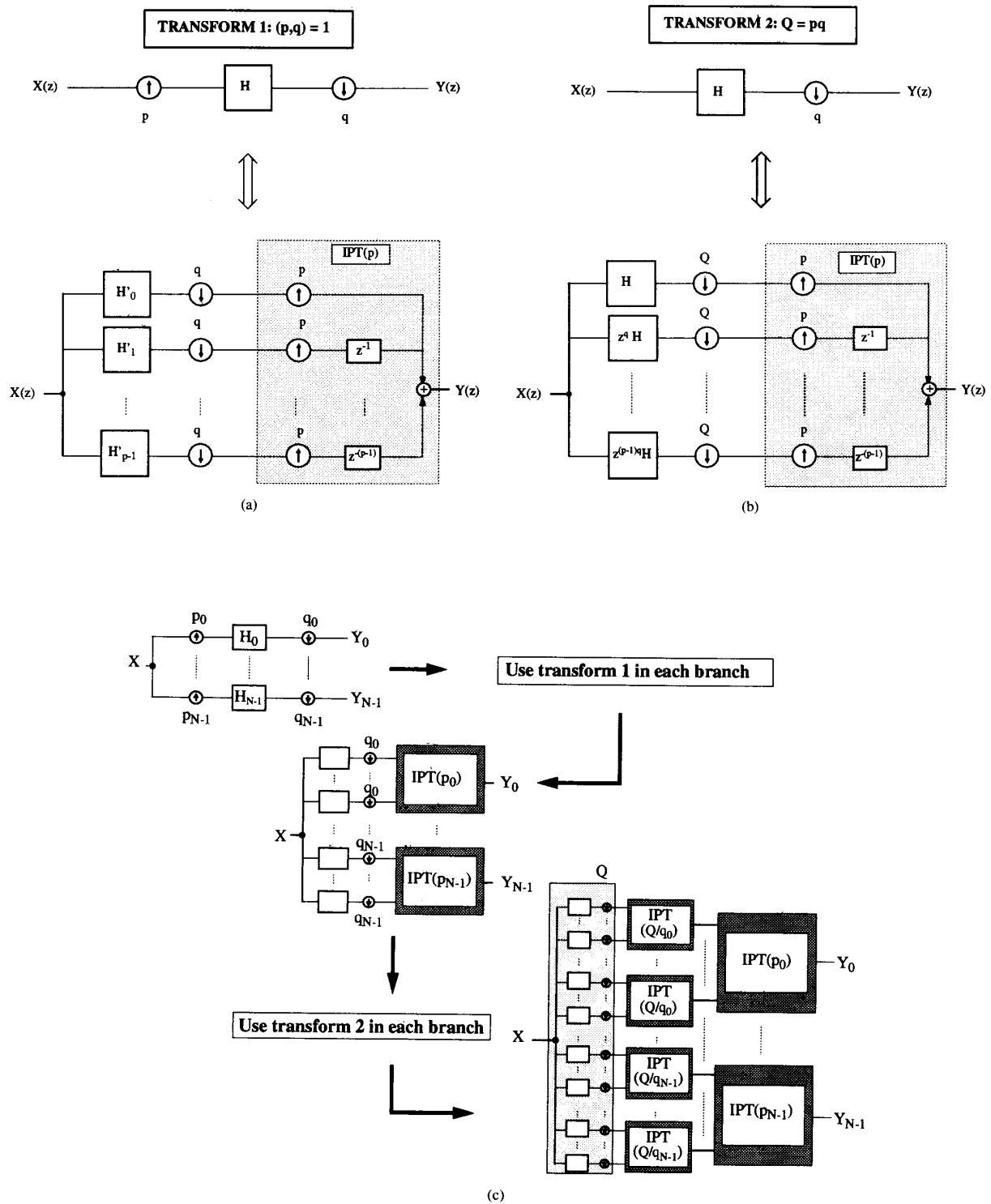


Fig. 10. (a) Transform 1: expressing a single branch with upsampling by  $p$  and downsampling by  $q$  using a  $p$ -channel analysis bank with sampling by  $q$  and an inverse polyphase transform of size  $p$ . All the filters involved are just shifted polyphase components of the original filter. For  $p = 1$  there is no transform. Also,  $p$  and  $q$  are assumed to be coprime. (b) Transform 2: expressing a single branch with downsampling by  $q$  using a  $p$ -channel analysis bank with sampling by  $Q = pq$  and an inverse polyphase transform of size  $p$ . All the filters involved are just shifted versions of the original filter. Note the dependency that appears in the filter banks after Transform 2. (3) To transform any bank we first apply Transform 1 and then Transform 2 in each branch. As a result an analysis bank with sampling by  $Q = \text{lcm}(q_0, \dots, q_{N-1})$  and  $Q$  branches is obtained.

with downsampling by  $Q = pq$  and an inverse polyphase transform of size  $p$ . The method is given in Fig. 10(b). The filter in the  $i$ th branch is just a shifted version of the original filter

$$H_i(z) = z^{iq} H(z). \quad (34)$$

An algebraic proof that the two representations in Fig. 10(b) are equivalent can be found in [13]. Here, a simpler, graphic proof, due to an anonymous reviewer, is outlined. In Fig. 10(b), move the filter  $H$  out of each branch and represent downsampling by  $Q$ , as downsampling by  $q$  followed by downsampling by  $p$ . Then, using the noble identities, one can move downsampling by  $q$  in front of delays, causing downsampling of the delays by  $q$ . The resulting system is then as follows: filter  $H$  followed by downsampling by  $q$ , followed by an identity system, i.e., the starting scheme.

Finally, Fig. 10(c) shows how by using the above transforms one can implement a filter bank from Fig. 2(a). First Transform 1 is applied in each branch which yields an analysis bank with  $n = \sum_{i=0}^{N-1} p_i$  branches and sampling factors  $q_0, \dots, q_{N-1}$ . Now if  $Q = \text{lcm}(q_0, \dots, q_{N-1})$  we apply Transform 2 in each branch to obtain an analysis bank with

$$n = \sum_{i=0}^{N-1} p_i \cdot \frac{Q}{q_i} = Q \sum_{i=0}^{N-1} \frac{p_i}{q_i} = Q$$

branches and downsampling by  $Q$ .

Note first that, as we said before, the case we want to implement has to be realizable. Then, if only Transform 1 is needed, all traditional methods for finding perfect reconstruction structures can be used. However, if Transform 2 has to be used as well, the problem becomes more complicated since this transform introduces dependencies among filters, unknown in the theory of uniform filter banks. This issue is left for future work.

It is also worth noting here the difference between the indirect and the direct methods (when they both work). In the indirect one, the two stages of the analysis bank are designed separately and moreover the characteristics of the equivalent filters ( $H_0, \dots, H_{N-1}$  from Fig. 2(a)) are unknown since we do not know how these filters are related to the filters in the analyzing and resynthesizing banks. Using a direct method however, allows one to design a filter bank with rational sampling factors, having at the same time the complete control over the desired characteristics of the filters  $H_0, \dots, H_{N-1}$ .

## V. CLASSIFICATION OF FILTER BANKS

In this section, realizable filter banks (as discussed in Section III) are divided into four classes. The first three, apart from being designed using a direct method (with more freedom in filter design) can be obtained using either partial solution from [5] or tree-splitting schemes or both (with more restricted filters). The fourth one however, can only be obtained using the method described in this paper.

To start, recall that in the filter bank we want to design

(see Fig. 2(a)) all the pairs  $(p_i, q_i)$  are coprime. The classification is then as follows (see also Fig. 11).

### A. Class 1: The $(2/3, 1/3)$ Case

This class contains all filter banks that could be designed using the partial solution from [5]. Equation (24) gives a necessary condition, together with the two facts that follow. Thus, the case  $(3/7, 1/7, 3/7)$  would belong to this class, but  $(3/7, 3/7, 1/7)$  would not. In Fig. 11, these facts are used to test whether a particular filter bank belongs to this class. A representative is the case with sampling factors  $(2/3, 1/3)$ . Note here that if the partial solution is used, filters obtained will be of the form (26), while using the direct method does not necessarily impose a particular structure, leading possibly to other solutions.

To construct this case, one could now choose one of the following methods: 1) use partial solution obtained in [5], yielding a low-pass of the form (27) or 2) use the direct method as given in Fig. 9(a).

Note that in the first one, one would have to design two banks, a 3-channel analysis, and a 2-channel synthesis, while in the second one, only a 3-channel analysis bank has to be constructed. In the first approach, one would use (27) and perform optimization on the low-pass and high-pass filters directly. In the second approach, a perfect reconstruction structure for a 3-channel analysis bank could be taken and then the magnitude responses of the two filters would be optimized. The structure should build a polyphase matrix that for this case would be

$$\mathbf{H}_p = \begin{pmatrix} H_{l0} & H_{l2} & H_{l4} \\ H_{l3} & H_{l5} & zH_{l1} \\ H_{h0} & H_{h1} & H_{h2} \end{pmatrix} \quad (35)$$

where  $H_{l0}, \dots, H_{l5}$  are the polyphase components of the low-pass filter with respect to 6, i.e., the low-pass filter is given by

$$H_l(z) = H_{l0}(z^6) + z^{-1}H_{l1}(z^6) + z^{-2}H_{l2}(z^6) + z^{-3}H_{l3}(z^6) + z^{-4}H_{l4}(z^6) + z^{-5}H_{l5}(z^6) \quad (36)$$

and  $H_{h0}, H_{h1}, H_{h2}$  are the polyphase components of the high-pass filter with respect to 3, i.e., the high-pass filter is given by

$$H_h(z) = H_{h0}(z^3) + z^{-1}H_{h1}(z^3) + z^{-2}H_{h2}(z^3). \quad (37)$$

In both approaches, specific structures should be chosen so as to obtain filters with prescribed properties, e.g., orthogonality or linear phase.

### B. Class 2: The $(1/2, 1/4, 1/4)$ Case

This class consists of all filter banks that could be built using tree-splitting schemes. Again, if tree splitting is used, the filters obtained are products of individual filters (possibly upsampled), while if the direct method is used, other solutions might be possible. A necessary condition

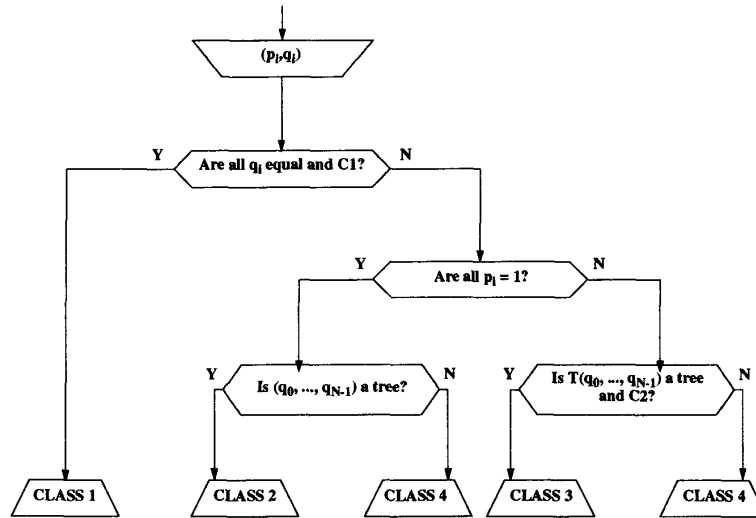


Fig. 11. Algorithm to perform classification of filter banks. Conditions C1 are given in Section III-D, while conditions C2 are given in Section V.

for a tree structure is given in (28), and the algorithm for testing whether the set of downsampling factors  $(q_0, \dots, q_{N-1})$  represents a tree is indicated in Fig. 8. A representative of this class is the case with sampling factors  $(1/2, 1/4, 1/4)$ .

#### C. Class 3: The $(2/3, 1/6, 1/6)$ Case

This class consists of all filter banks that can be built through combination of tree-splitting schemes and partial solutions (bearing in mind, however, that the direct approach might give more freedom in design). To be able to do that note that if one discards the synthesis banks (corresponding to recombinations), the rest still has to be a tree. Based on this define

$$T(q_0, \dots, q_{N-1}) = (q_0, \dots, q_0, \dots, q_{N-1}, \dots, q_{N-1}). \quad (38)$$

$\underbrace{\hspace{1.5cm}}_{p_0 \text{ times}} \qquad \underbrace{\hspace{1.5cm}}_{p_{N-1} \text{ times}}$

Define as conditions C2 conditions C1 from Section III-D applied on upsampling factors obtained from the first tree of  $T(q_0, \dots, q_{N-1})$ . For the case  $(2/3, 1/6, 1/6)$ ,  $T(3, 6, 6) = (3, 3, 6, 6)$  which yields the first 3-channel tree, and  $p_0 = 2, p_1 = 1, q = 3$ . Fig. 11 shows then that if not all downsampling factors are equal and if there is at least one upsampling factor greater than one, then testing whether  $T(q_0, \dots, q_{N-1})$  is a tree and C2 is satisfied, would yield filter banks belonging to this class. A representative is the case with sampling factors  $(2/3, 1/6, 1/6)$ .

#### D. Class 4: The $(3/7, 3/7, 1/7)$ Case

This class would then contain all filter banks that do not fit in the above three classes, i.e., all those that cannot be built using intermediate methods but have to be con-

structed using the direct design technique shown in this paper. A representative of this class is the case with sampling factors  $(3/7, 3/7, 1/7)$ . To design such a filter bank, the only possibility is to use the direct design method, i.e., design just one 7-channel analysis bank. Note that in view of the discussion given in Section III, this case can be constructed with ideal filters, and it will possess a polyphase transform. Designing a 7-channel analysis bank (leading to perfect reconstruction), one can cover the solutions in between these two extreme cases, as well.

## VI. DESIGN EXAMPLES

In this section a representative case, namely, sampling by  $2/3$  and  $1/3$  is going to be considered. Note that the case uses only Transform 1. The first three cases will give designs using the partial solution while the last two will use the direct design method. The filters obtained are of different lengths (low-pass of size 50 and high-pass of size 15 in cases 1, 2 and 3, low-pass of size 32 and high-pass of size 15 in cases 4 and 5).

#### A. Indirect Method versus Partial Solution

*Case 1:* As the first step let us construct such a filter bank indirectly (see Fig. 4(a)). Thus a 3-channel analysis filter bank and a 2-channel synthesis bank are needed. We use optimized lossless filter banks that already appeared in the literature in [11] and [14]. Since both banks are orthogonal the resulting filter bank is going to be orthogonal as well. To prove this fact, after some manipulations, the equivalent polyphase matrix becomes

$$\begin{pmatrix} G_p(z) \\ 1 \end{pmatrix} H_p(z) \quad (39)$$

where  $H_p(z)$ ,  $G_p(z)$  are the  $3 \times 3$  analysis and  $2 \times 2$  synthesis matrices, respectively. It is then trivial to check that the whole matrix is orthogonal if both  $H_p$  and  $G_p$  are orthogonal.

The 3-channel analysis filter bank is built using lossless building blocks consisting of delay matrices and constant unitary matrices. These unitary matrices are constructed as a sequence of planar rotations, and thus the angles involved are free variables when constructing the filter bank. The same is true for the 2-channel synthesis bank except that normalization can be performed simplifying the optimization process. For more details refer to [11], [14]. The 2-channel optimized bank (we will refer to it as PP2) contains filters of length 8 with lattice coefficients as given in Table I. The 3-channel optimized bank (we will refer to it as PP3) contains filters of length 15 with angles as given in Table II. Using tools that were developed in [5] and reviewed in Section III, the equivalent low-pass filter can be expressed as in (27). We remind the reader that the filters denoted by  $H_i$ ,  $i = 0, 1, 2$  belong to the 3-channel analysis bank and the ones denoted by  $G_i$ ,  $i = 0, 1$  belong to the 2-channel synthesis bank (see Fig. 4(a)). Thus the equivalent low-pass filter is now of length 50. Recall that this equivalent filter is located between upsampling by 2 and downsampling by 3 and thus when plotting its magnitude response one should bear in mind that it is actually plotted in the upsampled domain (thus if one wants a filter covering  $2/3$  of the spectrum, in the upsampled domain it would appear as a third-band filter). The high-pass filter is the third filter from the 3-channel analysis bank  $H_2$ . The magnitude response plots of the two filters are given in Fig. 12(a). As can be seen from the figure, the low-pass filter obtained in this fashion is of poor quality (see the response in the stopband).

**Case 2:** As the first step in analyzing this example, we want to see how well we can do if the optimization is performed directly on the low-pass filter as given in (27). For this case, the high-pass filter will be left the same, meaning that the optimization is done on the free variables of the 2-channel bank only (four of them). Ideally, one would like to have a low-pass with the transition band of  $2\epsilon$  in the original domain. Since the low-pass filter is being designed in the upsampled domain, the aim is to achieve as close a response to the one of a third-band filter with transition band of  $\epsilon$ . As a criterion for optimization, the error in the stopband will be evaluated, i.e.,

$$E = \int_{\pi/3 + \epsilon/2}^{\pi} |H_l(e^{jw})|^2 dw. \quad (40)$$

Note that in the lossless case, minimizing the error in the stopband automatically minimizes the error in the passband. To be able to compare this design to the one from Case 1, the same transition band as the one in [11], namely  $\epsilon = 0.382764898$  is used. The numerical optimization routine was taken from [15] and is based on a so-called "downhill simplex algorithm" in multidimensions. After running it with an extensive number of initial guesses, the

TABLE I  
CASE 1: COEFFICIENTS OF THE SEPARATELY OPTIMIZED 2-CHANNEL FILTER BANK FROM [14]

2-Channel Bank PP2	
$a_1$	-2.638026
$a_2$	0.7154463
$a_3$	-0.2598476
$a_4$	0.06388361

TABLE II  
CASE 1: COEFFICIENTS OF THE SEPARATELY OPTIMIZED 3-CHANNEL FILTER BANK FROM [11]

3-Channel Bank PP3			
$\alpha_1$	2.35919	$\beta_1$	-1.57089
$\alpha_2$	0.296676	$\beta_2$	-1.93762
$\alpha_3$	1.41305	$\beta_3$	3.14115
$\alpha_4$	-1.23644	$\beta_4$	-1.73044
$\alpha_5$	1.57091	$\beta_5$	2.35616

best local minimum obtained yields coefficients of the 2-channel bank as in Table III and filters (the high pass is the same as in Case 1) as in Fig. 12(b). As can be seen the improvement is obvious, the passband has been flattened and the stopband has been substantially reduced.

**Case 3:** In the previous case, the 3-channel bank has not been touched in order not to ruin the high-pass filter. The goal now is to try to optimize the filters using both filter banks. The optimization function is

$$E = \int_{\pi/3 + \epsilon/2}^{\pi} |H_l(e^{jw})|^2 dw + \int_0^{2\pi/3 - \epsilon} |H_2(e^{jw})|^2 dw. \quad (41)$$

The number of optimization variables is 14 (4 from the 2-channel and 10 from the 3-channel bank). As an educated guess we ran the optimization routine with coefficients as in Case 2 as initial values. After trying with other initial values, the first local minimum turned out to be the best. The coefficients of the optimized banks are given in Tables IV-V, and the filters' magnitude responses in Fig. 12(c). From the figures it is obvious that we have obtained a design that is even better than the previous one (compare the stopbands and the passbands). To achieve this improvement we have traded the sharpness of the transition band.

#### B. Using the Direct Design Method

**Case 4:** Let us first use the same 3-channel bank as in Case 1 and see how the equivalent filters look. Designing the 3-channel bank means designing a size  $3 \times 3$  lossless polyphase matrix which in turn yields filters  $H_0$ ,  $H_1$ , and  $H_2$ . Filter  $H_h = H_2$  is the high-pass filter we are interested in while the equivalent low-pass is given by (see Fig. 9(a))

$$H_l(z) = H_0(z^2) + z^{-1}H_1(z^2) \quad (42)$$

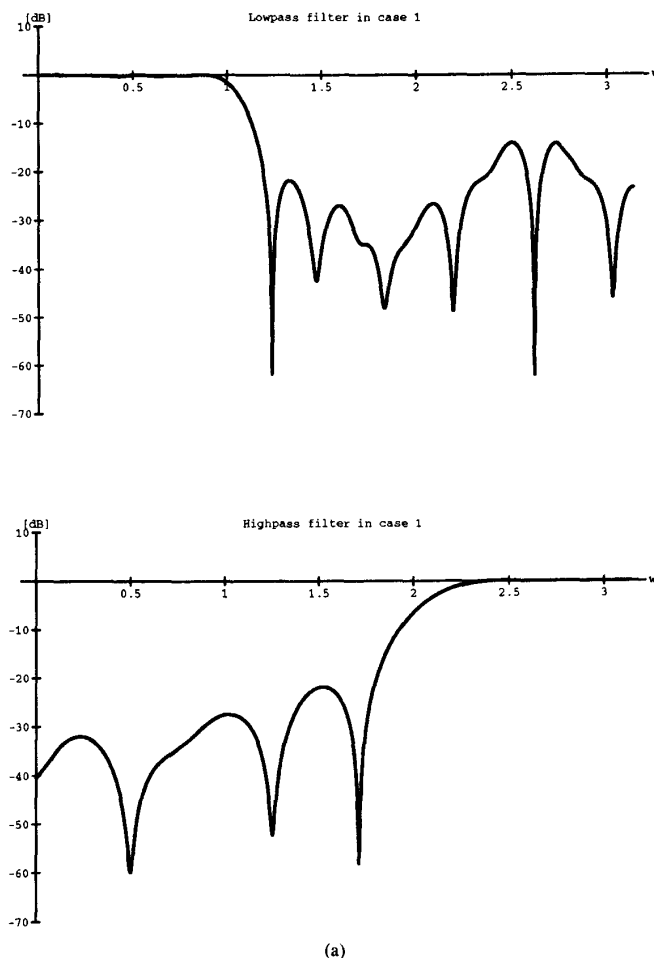


Fig. 12. (a) Case 1: Magnitude response plots of the filters obtained when separately optimizing two banks. (Continued on next page.)

where  $H_{l0} = H_0$  and  $H_{l1} = z^{-1}H_1$ . The high-pass filter is still of length 15 (it is the same as in Case 1) while the low-pass one is of length 32. The results are far from acceptable. The magnitude responses are given in Fig. 13(a) (high-pass is the same as in Case 1).

**Case 5:** Thus we now perform optimization directly on the magnitude responses of the two filters. The optimization criterion is the same as in (41). The final result is given in Table VI, while the magnitude responses are given in Figure 13(b). The insight and power obtained by using the direct design method are obvious, just compare the magnitude responses in Cases 4 and 5.

## VII. CONCLUSIONS AND FURTHER DIRECTIONS

The paper generalizes the theory of perfect reconstruction filter banks to the rational case thereby allowing for nonuniform division of the spectrum, a feature that could be useful in speech and music analysis (see [13]). The

commonly used filter banks with integer sampling factors thus become a special case. It was shown that in some cases various problems might arise, such as mirroring, reshuffling of frequencies, permutation of the bands, and can lead to nonrealizable systems. The first such system was pointed out in [2], leading to the notion of "compatible sets." Assuming realizable cases, the tools enabling the direct design method were developed, the first of which leads to structures where traditional methods from the theory of uniform filter banks can be used. The second one, however, introduces dependencies in the system, making the design much more involved. The development of structures tuned to Transform 2 is left for future work. The classification of filter banks was then performed and the discussion of particular structures was presented. Several design examples were given.

An interesting direction for future work might be to extend this theory to the multidimensional case. Instead of sampling factors one would deal with sampling lattices.

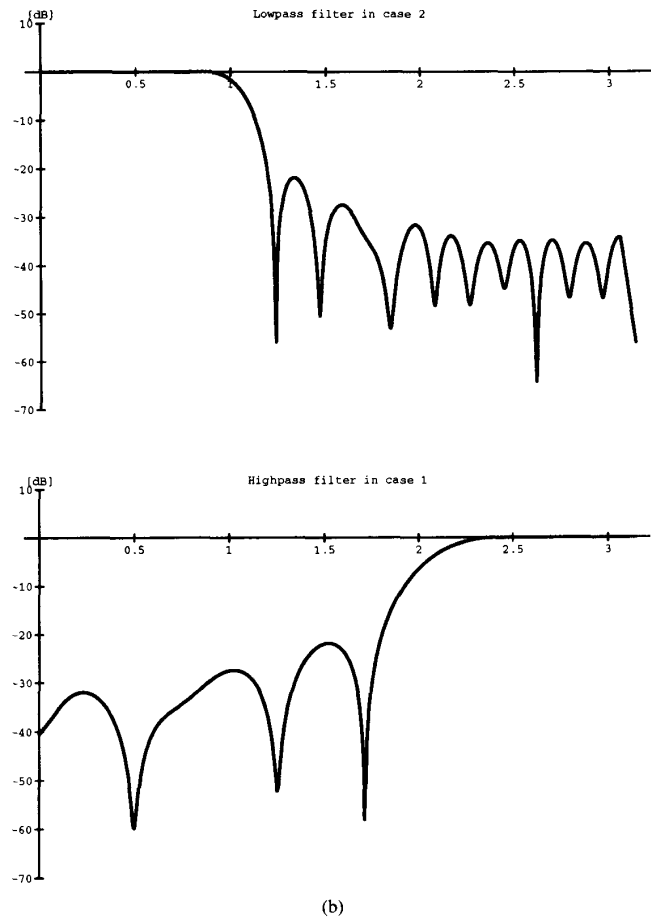


Fig. 12. (Continued.) (b) Case 2: Magnitude response plots of the filters obtained when optimizing the 2-channel bank without touching the 3-channel one. Therefore, the high-pass filter is the same as in Case 1. (Continued on next page.)

A necessary condition for critical sampling would then be

$$\sum_i \frac{\det(\mathbf{D}_{1i})}{\det(\mathbf{D}_{2i})} = 1$$

where  $\mathbf{D}_{1i}$ 's represent upsampling lattices and  $\mathbf{D}_{2i}$ 's represent downsampling lattices. Since Transform 1 depends on the interchangeability of up/downsampling, a key ingredient needed would be the commutativity of up/downsampling that appeared in [7].

Note also, that due to recently developed connection between wavelet theory and filter banks, there exist a number of questions of interest concerning the connection in the present rational case, some of which are addressed in [13].

#### APPENDIX A

##### TREE-SPLITTING VERSUS DIRECT DESIGN METHOD: THE LOSSLESS CASE FOR $N = 2$

This Appendix aims to show that if the system obtained by tree splitting for the  $(1/2, 1/4, 1/4)$  case is lossless, then the system in Fig. 9(b) is lossless as well. In other

words, all lossless solutions obtained by tree-splitting schemes can be generated using the direct design method as expected. Assume that for the tree-splitting scheme the first filter bank has filters denoted by  $H_0$  and  $H_1$ , while the second one has filters  $T_0$  and  $T_1$ . Note that the two fourth-band filters (in Fig. 9(b)) can be expressed as  $H_1(z)T_0(z^2)$  and  $H_1(z)T_1(z^2)$  (just by moving filters  $T_0$  and  $T_1$  across downsamplers). Thus, after some manipulations, the polyphase matrix in Fig. 9(b), that is, the polyphase matrix of the expanded analysis bank in Fig. 9(b), can be expressed in terms of filters from the tree-splitting scheme as

$$\mathbf{H}_p = \begin{pmatrix} \mathbf{I} & \\ & \mathbf{T}_p \end{pmatrix} \begin{pmatrix} H_0 & z\mathbf{D}H_0 \\ \mathbf{D}H_1 & \mathbf{J}H_1 \end{pmatrix} = \mathbf{T} \cdot \mathbf{H} \quad (43)$$

where  $\mathbf{T}_p$  is a lossless matrix containing polyphase components of filters  $T_0$  and  $T_1$  and thus the whole matrix  $\mathbf{T}$  is lossless. Moreover, since  $\mathbf{T}$  is lossless the only fact left to prove is that  $\mathbf{H}$  is lossless, since the cascade of two lossless matrices would give a lossless  $\mathbf{H}_p$ . In (43),  $\mathbf{J}$ ,  $\mathbf{D}$ ,

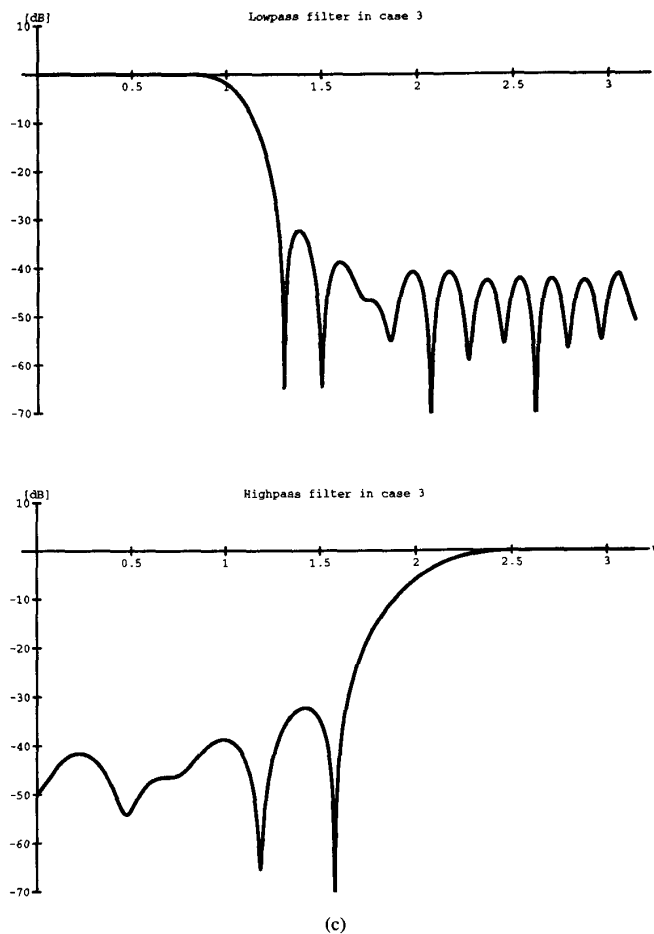


Fig. 12. (Continued.) (c) Case 3: Magnitude response plots of the filters obtained when optimizing both banks at the same time.

TABLE III  
CASE 2: COEFFICIENTS OF THE OPTIMIZED 2-CHANNEL FILTER BANK WHEN  
THE 3-CHANNEL BANK IS NOT TOUCHED

2-Channel Bank OP2a	
$a_1$	-0.371151
$a_2$	2.732850
$a_3$	1.056070
$a_4$	0.664108

TABLE IV  
CASE 3: COEFFICIENTS OF THE 2-CHANNEL FILTER BANK WHEN BOTH THE 2-  
CHANNEL AND 3-CHANNEL BANKS ARE OPTIMIZED AT THE SAME TIME

2-Channel Bank OP2b	
$a_1$	-0.513454
$a_2$	3.072208
$a_3$	1.566488
$a_4$	0.305714

TABLE V  
CASE 3: COEFFICIENTS OF THE 3-CHANNEL FILTER BANK WHEN BOTH THE  
2-CHANNEL AND 3-CHANNEL FILTER BANKS ARE OPTIMIZED  
AT THE SAME TIME

3-Channel Bank OP3b			
$\alpha_1$	2.353882	$\beta_1$	-1.537590
$\alpha_2$	0.152418	$\beta_2$	-1.845041
$\alpha_3$	1.57162	$\beta_3$	3.212307
$\alpha_4$	-1.213656	$\beta_4$	-1.682972
$\alpha_5$	1.251406	$\beta_5$	2.375487

TABLE VI  
CASE 5: COEFFICIENTS OF THE OPTIMIZED 3-CHANNEL FILTER BANK

3-Channel Bank OP3c			
$\alpha_1$	1.848559	$\beta_1$	-2.004359
$\alpha_2$	0.580876	$\beta_2$	-1.752920
$\alpha_3$	1.199016	$\beta_3$	2.672730
$\alpha_4$	-1.385383	$\beta_4$	-1.574727
$\alpha_5$	3.005676	$\beta_5$	1.957543



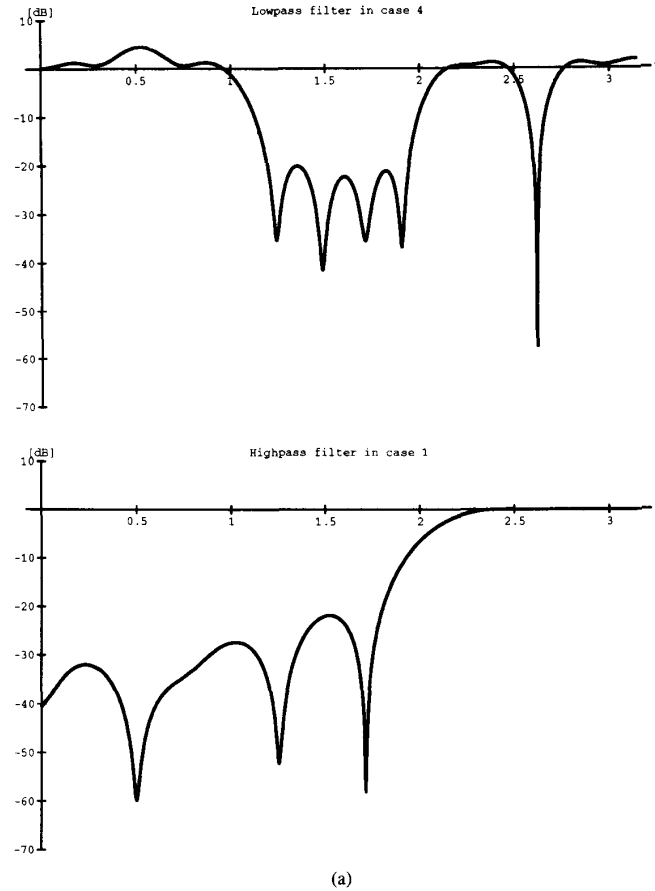


Fig. 13. (a) Case 4: Magnitude response plots of the filters obtained when using just the 3-channel bank from [14]. Therefore, the high-pass filter is the same as in Case 1. (Continued on next page.)

$H_0$  and  $H_1$  denote the following:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (44)$$

$$H_0 = \begin{pmatrix} H_{00} & H_{01} \\ H_{02} & H_{03} \end{pmatrix}, \quad H_1 = \begin{pmatrix} H_{10} & H_{11} \\ H_{12} & H_{13} \end{pmatrix} \quad (45)$$

where  $H_{ij}$  are the four polyphase components of the first and second filters. We prove by induction that  $H$  is lossless. The fact that all lossless polyphase matrices can be built using the following cascade structure [14] is going to be used (superscript (2) stands for polyphase decomposition with respect to 2):

$$H_{p(k+1)}^{(2)} = c H_{pk}^{(2)} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} \quad (46)$$

with

$$H_{p0}^{(2)} = c_0 \begin{pmatrix} 1 & \beta \\ -\beta & 1 \end{pmatrix} \quad (47)$$

where  $c_0$  is the appropriate normalization.

*Initial step:* Substituting the values for polyphase components from (47) into matrix  $H$  yields

$$H = \begin{pmatrix} 1 & \beta & 0 & 0 \\ 0 & 0 & z & \beta z \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & -\beta & 1 \end{pmatrix} \quad (48)$$

for which it is trivial to check that it is lossless.

*Inductive step:* Suppose now that the assumption holds for  $k$  and we want to prove it for  $(k+1)$ . Thus assume that the matrix  $H$  obtained from

$$H_{pk}^{(2)} = \begin{pmatrix} H_{00}^{(2)}(z) & H_{01}^{(2)}(z) \\ H_{10}^{(2)}(z) & H_{11}^{(2)}(z) \end{pmatrix} = \begin{pmatrix} H_{00}(z^2) + z^{-1}H_{02}(z^2) & H_{01}(z^2) + z^{-1}H_{03}(z^2) \\ H_{10}(z^2) + H_{12}(z^2) & H_{11}(z^2) + z^{-1}H_{13}(z^2) \end{pmatrix} \quad (49)$$

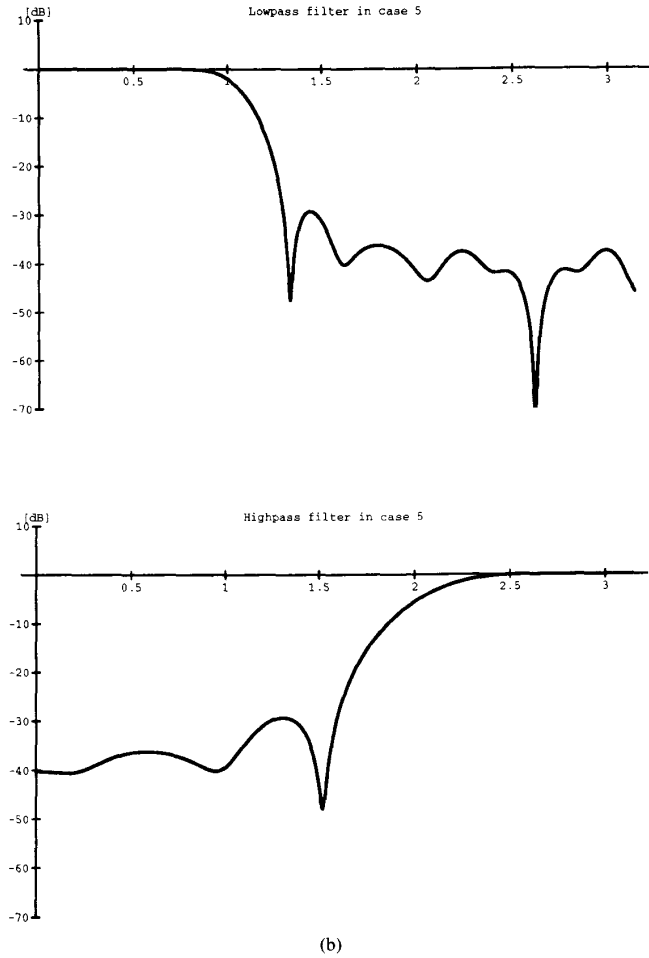


Fig. 13. (Continued.) (b) Case 5: Magnitude response plots of the filters obtained when optimizing directly the 3-channel bank.

is lossless. Then if one forms the next step in the cascade as in (46), after some manipulations it becomes obvious that the corresponding matrix  $\mathbf{H}$  is lossless as well.  $\square$

#### APPENDIX B PROOFS OF PROPOSITIONS FROM SECTION III

##### 1. Proof of Proposition 3.1

Suppose the statement of the proposition is not true, i.e.,  $H$  is positioned at

$$H: \left[ \left( \frac{s}{q} + \epsilon \right) \pi, \left( \frac{s+1}{q} + \epsilon \right) \pi \right] \quad (50)$$

and symmetrically for negative frequencies. The closest replica of the negative part of the spectrum is at

$$H_L: \left[ 2\pi - \left( \frac{s+1}{q} + \epsilon \right) \pi, 2\pi - \left( \frac{s}{q} + \epsilon \right) \pi \right]. \quad (51)$$

After down/upsampling, aliased version  $H(\omega + 2i\pi/q)$  will move  $H_L$  to

$$H'_L: \left[ 2\pi - \left( \frac{s+1}{q} + \epsilon + \frac{2i}{q} \right) \pi, 2\pi - \left( \frac{s}{q} + \epsilon + \frac{2i}{q} \right) \pi \right]. \quad (52)$$

It suffices to show now that one of the frequencies belonging to  $H'_L$  will coincide with one in  $H$ . Thus, let us check that the following is true:

$$2\pi - \left( \frac{s+1}{q} + \frac{2i}{q} \right) \pi = \frac{s+1}{q} \pi \quad (53)$$

or

$$q = s + i + 1. \quad (54)$$

But since  $s, i = 0, \dots, q-1$ , there always exists  $i$  so that (54) is true, or, in other words,  $H'_L$  will overlap with  $H$  causing aliasing and disabling reconstruction.  $\square$

## 2. Proof of Proposition 3.5

We will show this for just one analyzing/synthesizing part, that is, for  $p$  analysis filters  $H_k, \dots, H_{k+p-1}$  with  $k$  even if  $p > 1$ , followed by downsampling by  $q$ , upsampling by  $p$  and  $p$  synthesis filters  $G_0, \dots, G_{p-1}$ . The first analysis filter  $H_k$  is positioned from

$$H_k: \left[ \frac{k}{q} \pi, \frac{k+1}{q} \pi \right]$$

in the spectrum, while  $G_i$  resides at

$$G_i: \left[ \frac{i}{p} \pi, \frac{i+1}{p} \pi \right].$$

Upsampled versions will then be at

$$\begin{aligned} H_k(z^p): & \left[ \frac{k+2nq}{pq} \pi, \frac{k+1+2nq}{pq} \pi \right] \\ \cup & \left[ \frac{-k+(2n-1)q}{pq} \pi, \frac{-k-1+(2n-1)q}{pq} \pi \right] \end{aligned} \quad (55)$$

$$\begin{aligned} G_0(z^q): & \left[ \frac{2mp}{pq} \pi, \frac{2mp+1}{pq} \pi \right] \\ \cup & \left[ \frac{(2m-1)p}{pq} \pi, \frac{(2m-1)p-1}{pq} \pi \right] \end{aligned} \quad (56)$$

with  $k = 0, 2, \dots, k \leq q-p, n = 0, \dots, p-1, m = 0, \dots, q-1$ . Thus, the first filter from (26)  $H_k(z^p)G_0(z^q)$  will live where the above two intersect. The solution is the solution of the following equation:

$$k + 2nq = 2mp. \quad (57)$$

Since  $k$  is even, and  $(p, q)$  are coprime, (57) has a solution. Therefore, the right-hand side of the equivalent filter from (26)  $H_e$  will be at

$$H_{ek}: \left[ \frac{k+2nq}{pq} \pi, \frac{k+p+2nq}{pq} \pi \right] \quad (58)$$

if  $0 \leq k+p+2nq \leq pq$ , or from

$$H_{ek}: \left[ \frac{2pq-k-p-2nq}{pq} \pi, \frac{2pq-k-2nq}{pq} \pi \right] \quad (59)$$

if  $pq \leq k+p+2nq \leq 2pq$ . First, we have to show that the filter obtained is a correct  $q$ th band filter as required by Proposition 3.1. Substituting (57) into both (58) and (59), one obtains

$$H_{ek}: \left[ \frac{2m}{q} \pi, \frac{(2m+1)}{q} \pi \right], \quad 0 \leq k+p+2nq \leq pq \quad (60)$$

$$\begin{aligned} H_{ek}: & \left[ \frac{(2q-2m-1)}{q} \pi, \frac{(2q-2m)}{q} \pi \right], \\ & pq \leq k+p+2nq \leq 2pq \end{aligned} \quad (61)$$

that is, a correct  $q$ th band filter.

Then, one has to show that this  $q$ th band filter lives within one "half-image" of the input signal as explained in Section III-C. For the first case (58) (for the second one similarly), substitute (57) into (17) where  $s = 2m$ , i.e.,

$$(2m+1)\frac{p}{q} - 1 \leq l \leq \frac{2mp}{q}.$$

For  $k = 0$  this obviously works, yielding  $l = 2n$ , while for  $k \neq 0$ , one gets  $k \leq q-p$ , which is satisfied by assumption, yielding  $l = 2n$  again.

The only thing left to show is that the part extracted from the signal is indeed correct. In the first case, with  $l = 2n$ , the part extracted is

$$\begin{aligned} & \left[ \left( 2m\frac{p}{q} - 2n \right) \pi, \left( (2m+1)\frac{p}{q} - 2n \right) \pi \right] \\ & = \left[ \frac{k}{q} \pi, \frac{k+p}{q} \pi \right] \end{aligned}$$

that is, the correct part of the input signal's spectrum. The second case follows similarly.  $\square$

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## REFERENCES

- [1] R. N. J. Veldhuis, M. Breeuwer, and R. G. Van der Waal, "Subband coding of digital audio signals," *Philips J. Res.*, vol. 44, no. 2/3, pp. 329-343, 1989.
- [2] P.-Q. Hoang and P. P. Vaidyanathan, "Nonuniform multirate filter banks: Theory and design," in *Proc. IEEE Int. Symp. Circuits Syst.*, Portland, OR, 1989, pp. 371-374.
- [3] M. J. T. Smith and T. P. Barnwell, III, "Exact reconstruction for tree-structured subband coders," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 34, pp. 431-441, June 1986.
- [4] K. Nayeibi, T. P. Barnwell, III, and M. J. T. Smith, "The design of perfect reconstruction nonuniform band filter banks," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Toronto, Canada, May, 1991, pp. 1781-1784.
- [5] J. Kovačević and M. Vetterli, "Perfect reconstruction filter banks with rational sampling rates in one and two dimensions," in *Proc. SPIE Conf. Visual Commun. Image Processing*, Philadelphia, PA, Nov. 1989, pp. 1258-1268.
- [6] J. Kovačević and M. Vetterli, "Perfect reconstruction filter banks with rational sampling rate changes," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Toronto, Canada, May 1991, pp. 1785-1788.
- [7] J. Kovačević and M. Vetterli, "The commutativity of up/downsampling in two dimensions," *IEEE Trans. Inform. Theory*, vol. 37, pp. 695-698, May 1991.
- [8] M. Vetterli, "A theory of multirate filter banks," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 35, pp. 356-372, Mar. 1987.
- [9] P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial," *Proc. IEEE*, vol. 78, pp. 56-93, Jan. 1990.
- [10] P. P. Vaidyanathan, "Quadrature mirror filter banks,  $M$ -band extensions and perfect reconstruction techniques," *IEEE ASSP Mag.*, vol. 4, pp. 4-20, July 1987.
- [11] P. P. Vaidyanathan, "Theory and design of  $M$ -channel maximally decimated quadrature mirror filters with arbitrary  $M$ , having the perfect reconstruction property," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 35, pp. 476-492, Apr. 1987.

- [12] C.-C. Hsiao, "Polyphase filter matrix for rational sampling rate conversion," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Dallas, TX, Apr. 1987, pp. 2173-2176.
- [13] J. Kovačević, "Filter banks and wavelets: Extensions and applications," Ph.D. dissertation, Columbia Univ., New York, NY, Oct. 1991.
- [14] P. P. Vaidyanathan and P.-Q. Hoang, "Lattice structures for optimal design and robust implementation of two-channel perfect reconstruction filter banks," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 81-94, Jan. 1988.
- [15] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C*. New York, Cambridge University Press, 1988.



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